Parallel Batch-Dynamic *k*-Core Decomposition

Julian Shun (MIT CSAIL)

Joint work with Quanquan Liu, Jessica Shi, Shangdi Yu, and Laxman Dhulipala

Announcements

- No class next Tuesday 11/11
- Mid-term report due Friday 11/14

Graphs are becoming very large

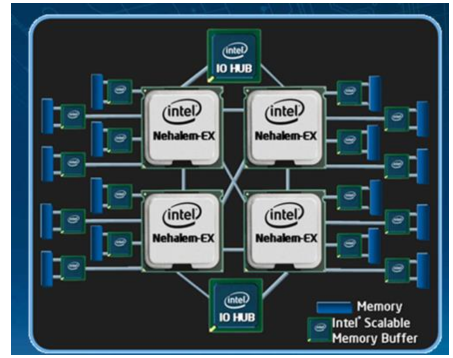
	Size	
Common Crawl	3.5 billion vertices 128 billion edges	*Largest publicly available graph*
YAHOO!	272 billion vertices 5.9 trillion edges	*Proprietary graph*
Google	> 100 billion vertices 6 trillion edges	*Proprietary graph*

Graphs are rapidly changing

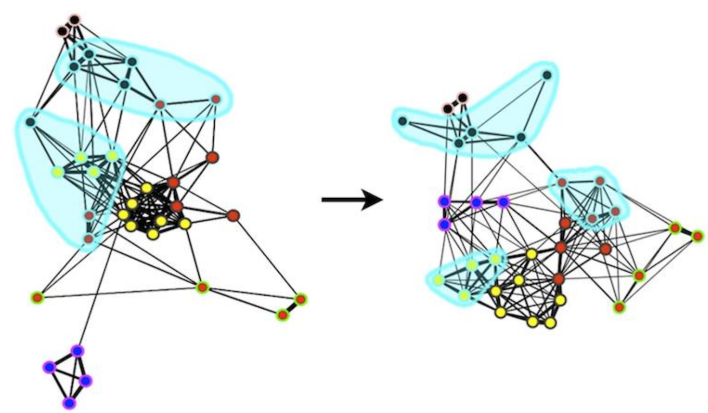
(500M tweets/day,
547K new websites/day)

Parallelism and Dynamic Algorithms for High Performance

- Take advantage of parallel machines

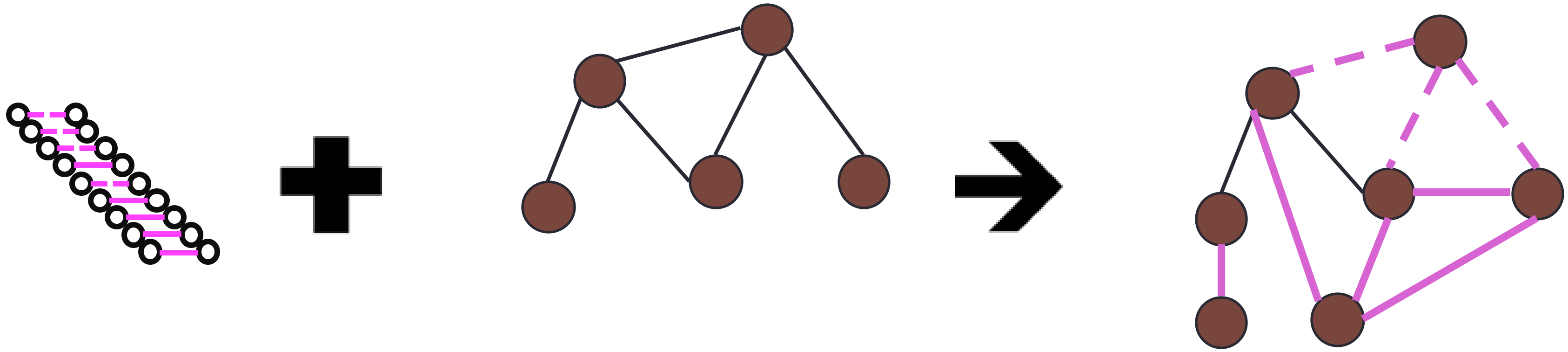


- Design dynamic algorithms to avoid unnecessary work on updates



Parallel Batch-Dynamic Algorithms

- Process updates in batches, and use parallelism within each batch



A **batch** of edge insertions/deletions

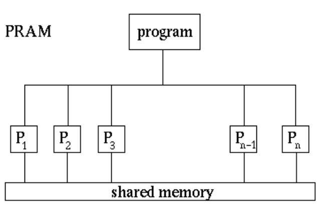
Current graph + Current statistics

Updated graph + Updated statistics

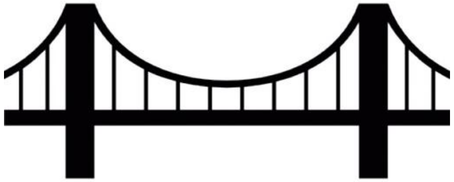
Insertion
Deletion

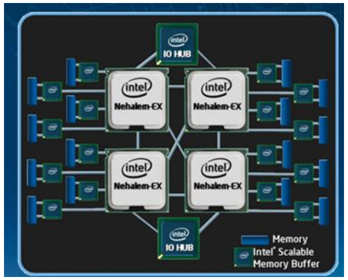
Our Parallel Batch-Dynamic Algorithms

k-core decomposition
Clique counting
Low out-degree orientation
Maximal matching
Graph coloring
Minimum spanning forest
Single-linkage clustering
Closest pair



O(n log n)
O(n)
O(log n)





Theory

Practice

Quanquan C. Liu, Jessica Shi, Shangdi Yu, Laxman Dhulipala, Julian Shun, *"Parallel Batch-Dynamic Algorithms for k-Core Decomposition and Related Graph Problems,"* SPAA 2022

Related Work on Parallel Batch-Dynamic Algorithms

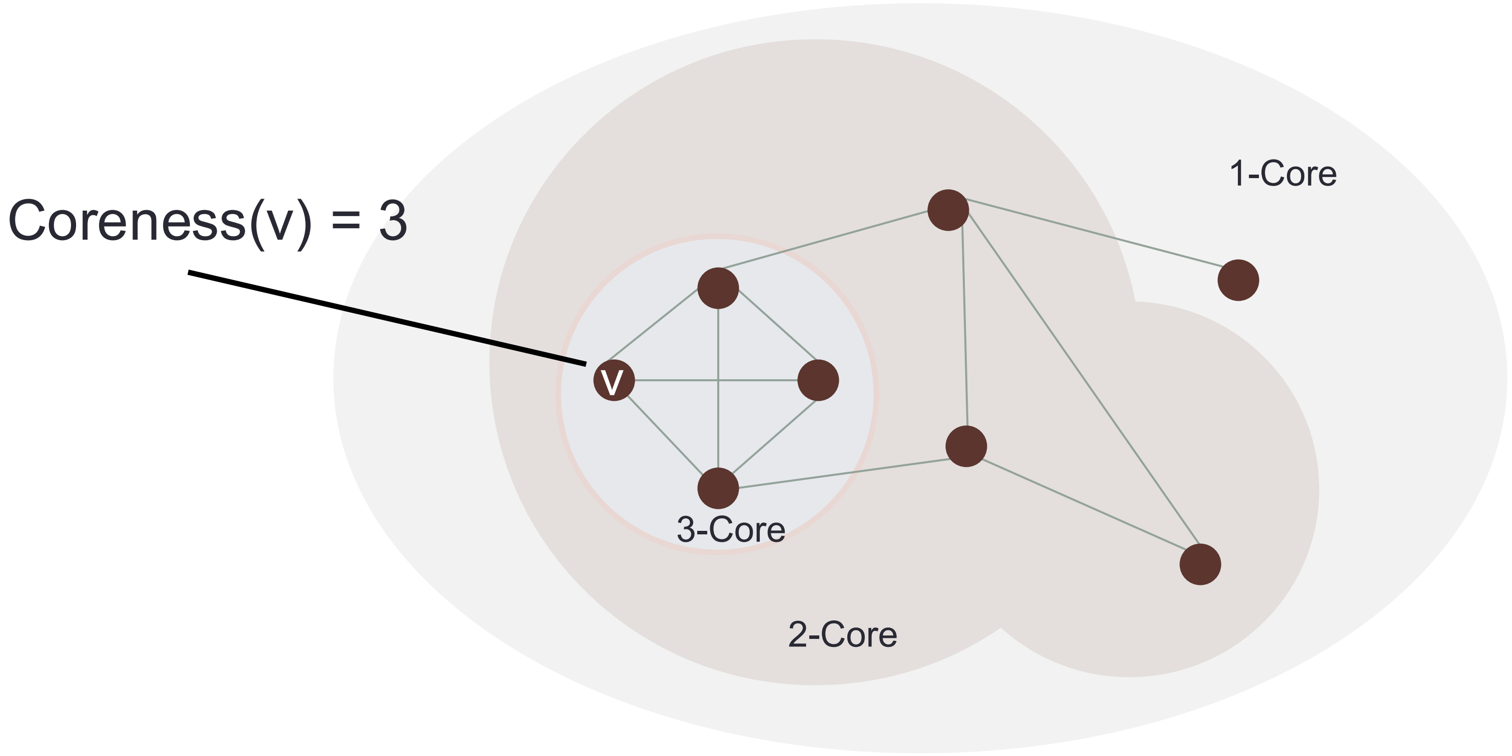
- Triangle Counting [EB10, MBG17]
- Euler Tour Trees [TDB19]
- Connected Components [FL94, MGB13, AABD19]
- Rake-Compress Trees [AABDW20]
- Incremental Minimum Spanning Trees [ABT20]

k-Core Decomposition

k-Core Decomposition

k-core: maximal connected subgraph of G such that all vertices have induced degree $\geq k$

Coreness(v): largest value of *k* where v participates in the *k*-core

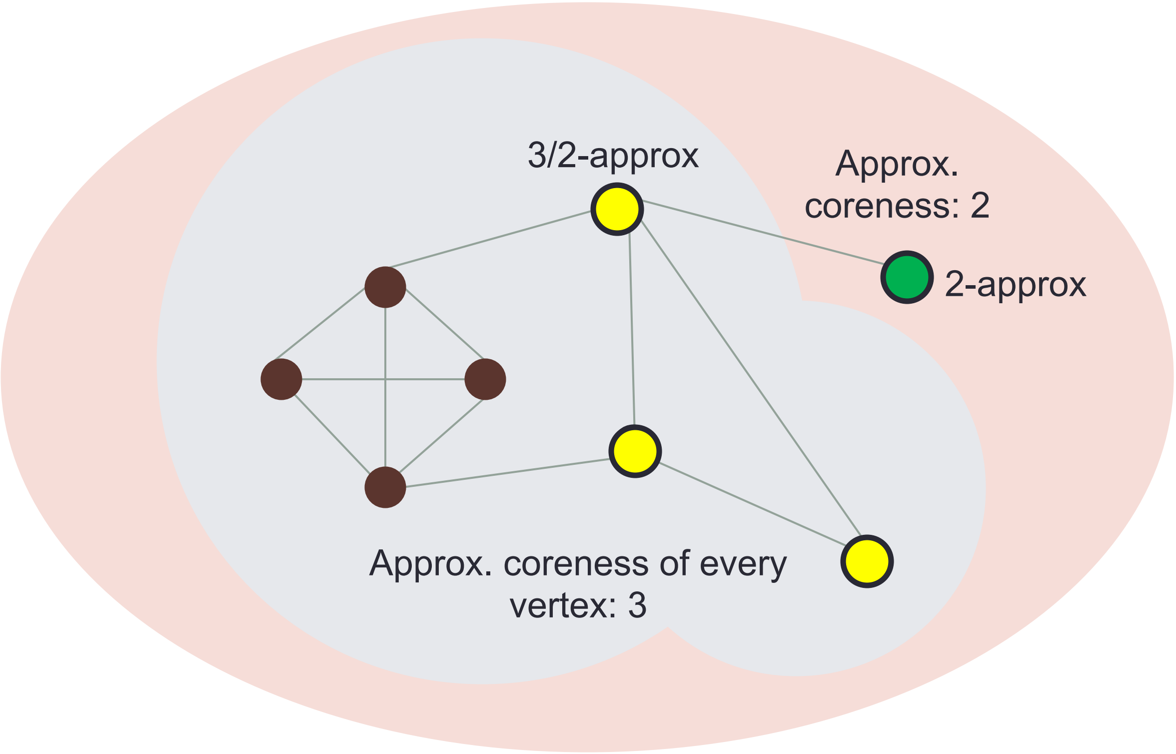


Goal: compute coreness for all vertices

Approximate *k*-Core Decomposition

k-core: maximal connected subgraph of G such that all vertices have induced degree $\geq k$

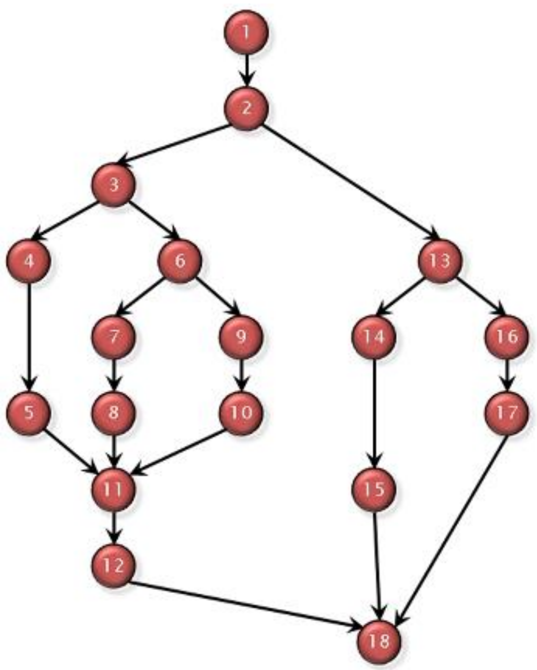
c-Approx-Coreness(v): value within multiplicative *c* factor of Coreness(v)



Applications of *k*-core Decomposition

- Graph clustering
- Community detection
- Graph visualization
- Protein network analysis
- Approximating network centrality

Work-Span Model



***Work** = number of operations*
***Span** = length of longest sequential dependence*

***Running time** ≤ (Work/#processors) + O(Span)*

- Goal: Design **low-span** parallel algorithms that are **work-efficient** (work asymptotically matches that of the best sequential algorithm)

Our Results for *k*-core Decomposition

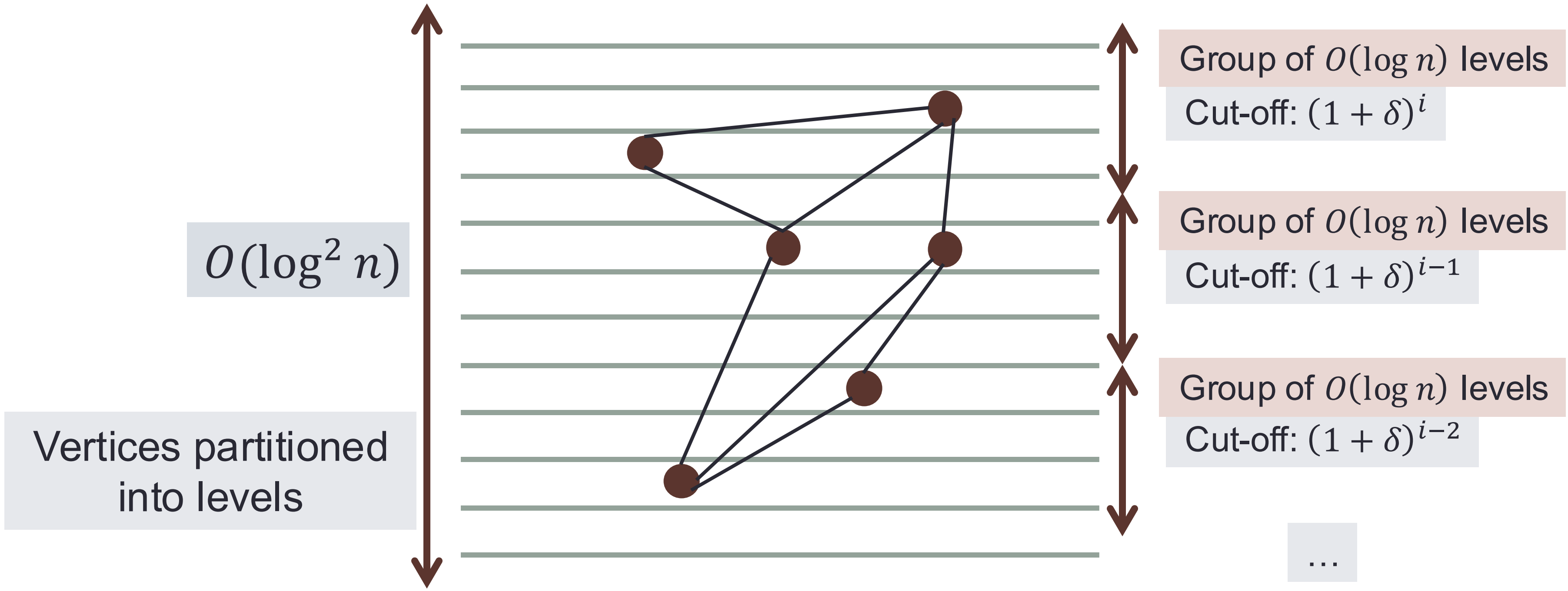
- Our algorithm dynamically maintains a $(\mathbf{2}+\boldsymbol{\epsilon})$**-approximation** for coreness of every vertex
- A batch of B updates takes $O(B \log^2 n)$ amortized work and $O(\log^2 n \log\log n)$ span with high probability
- Our algorithm is work-efficient, matching the work of the state-of-the-art sequential algorithm by Sun et al.
- Our algorithm is based on a **parallel level data structure**

Sequential Level Data Structures for Dynamic Problems

- Maximal Matching [Baswana-Gupta-Sen ‘18, Solomon ‘16]
- $(\Delta + 1)$-Coloring [Bhattacharya-Chakrabarty-Henzinger-Nanongkai ‘18, Bhattacharya-Grandoni-Kulkarni-Liu-Solomon ‘19]
- Clustering [Wulff-Nilsen ‘12]
- Low out-degree orientation [Solomon-Wein ‘20, Henzinger-Neumann-Weiss ‘20]
- Densest subgraph [Bhattacharya-Henzinger-Nanongkai-Tsourakakis ‘15]

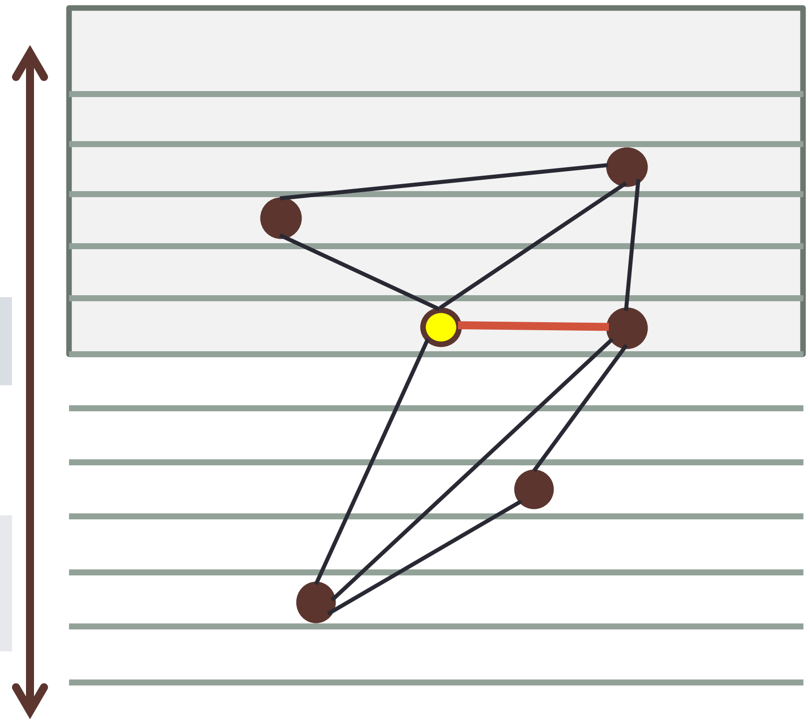
Sequential Level Data Structure (LDS)

- Described by Bhattacharya, Henzinger, Nanongkai, Tsourakakis [2015] and Henzinger, Neumann, Wiese [2020]



- Maintain invariants per vertex, which give upper/lower bounds on roughly its number of "**up-neighbors**" (neighbors at around its level and above)
- We prove that levels translate to coreness estimates

Sequential Level Data Structure (LDS)



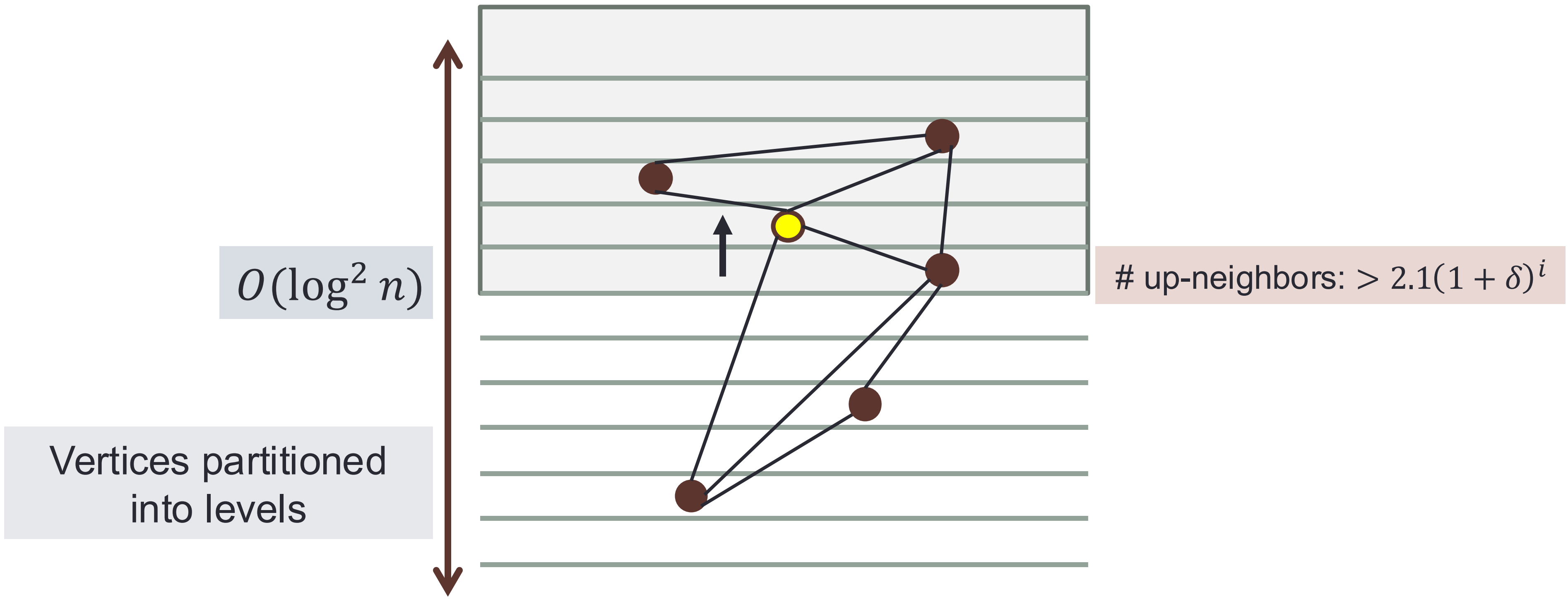
$O(\log^2 n)$

Vertices partitioned into levels

up-neighbors: $> 2.1(1+\delta)^i$

= edge insertion

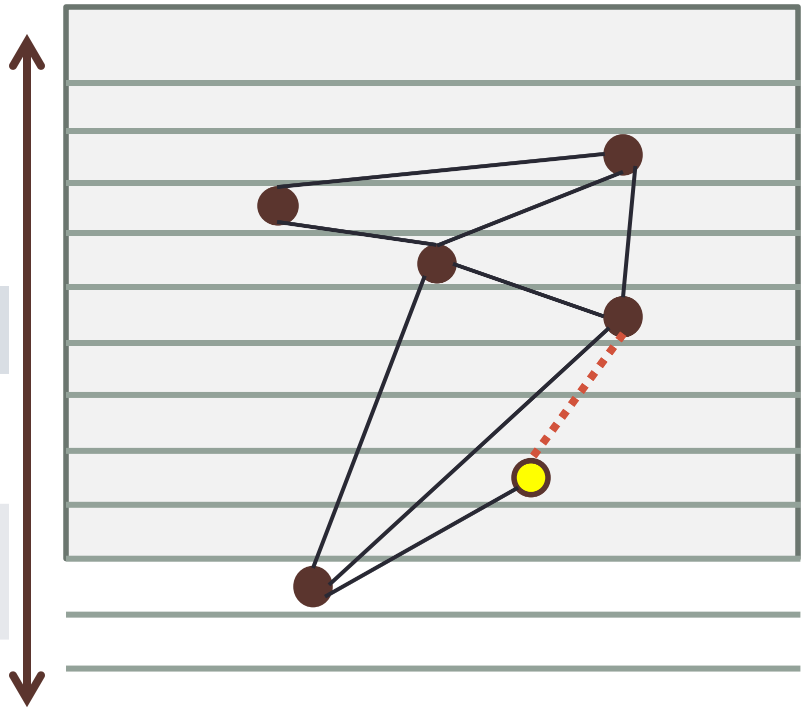
Sequential Level Data Structure (LDS)



Sequential Level Data Structure (LDS)

$O(\log^2 n)$

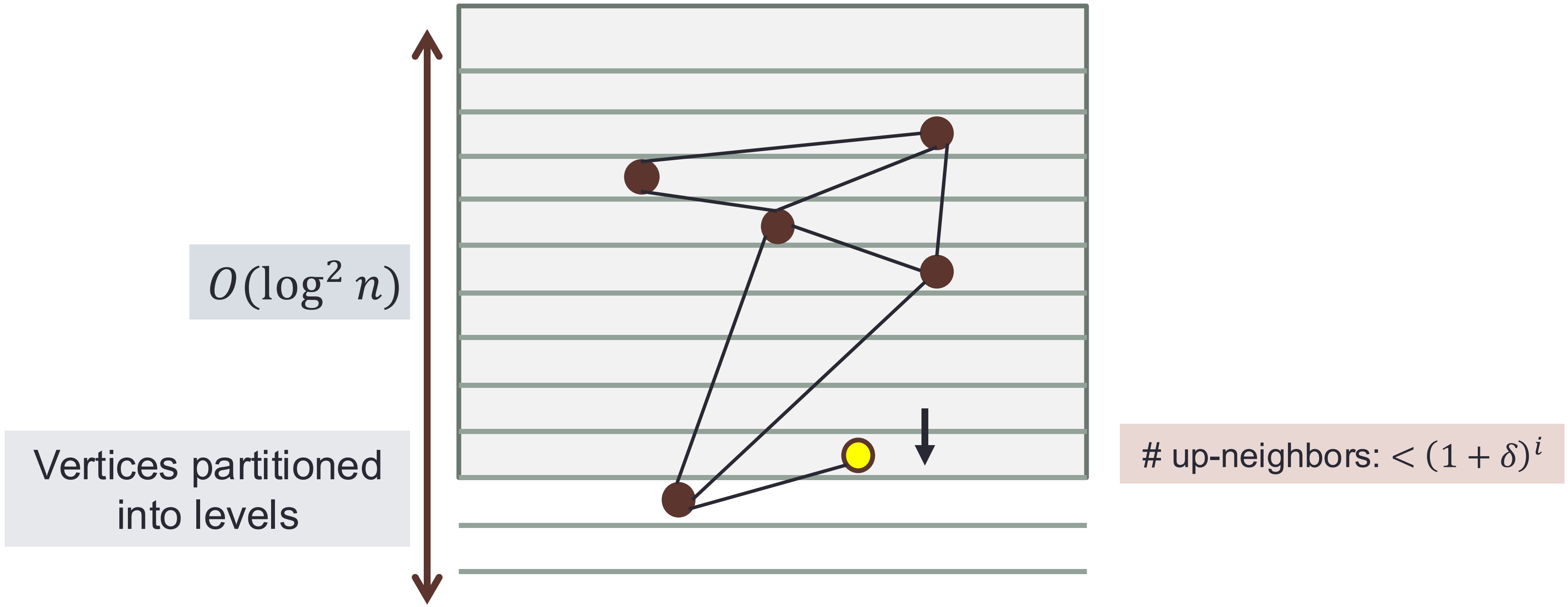
Vertices partitioned into levels



= edge deletion

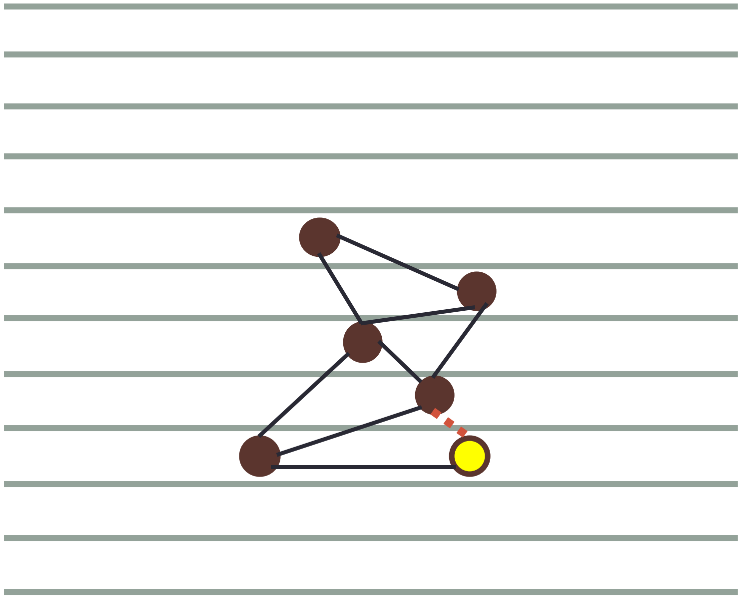
up-neighbors: $< (1+\delta)^i$

Sequential Level Data Structure (LDS)



Difficulties with Parallelization

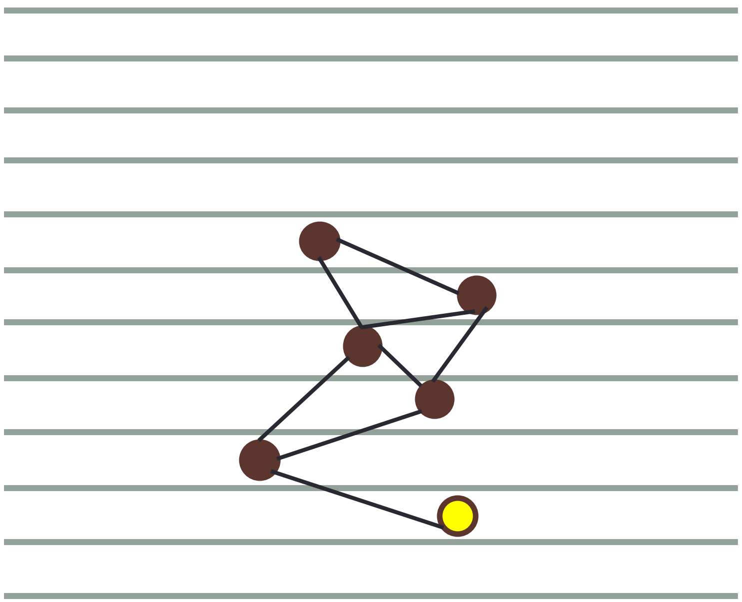
Large sequential dependencies



Large span

Difficulties with Parallelization

Large sequential dependencies



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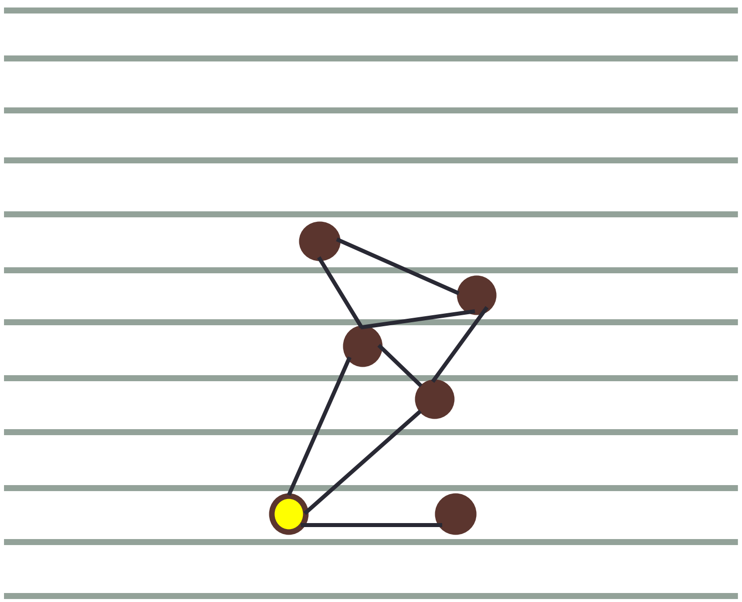

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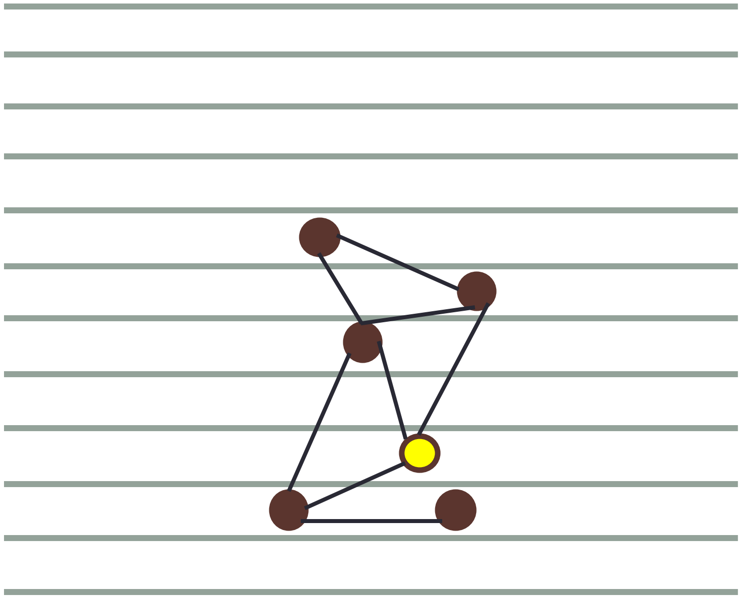

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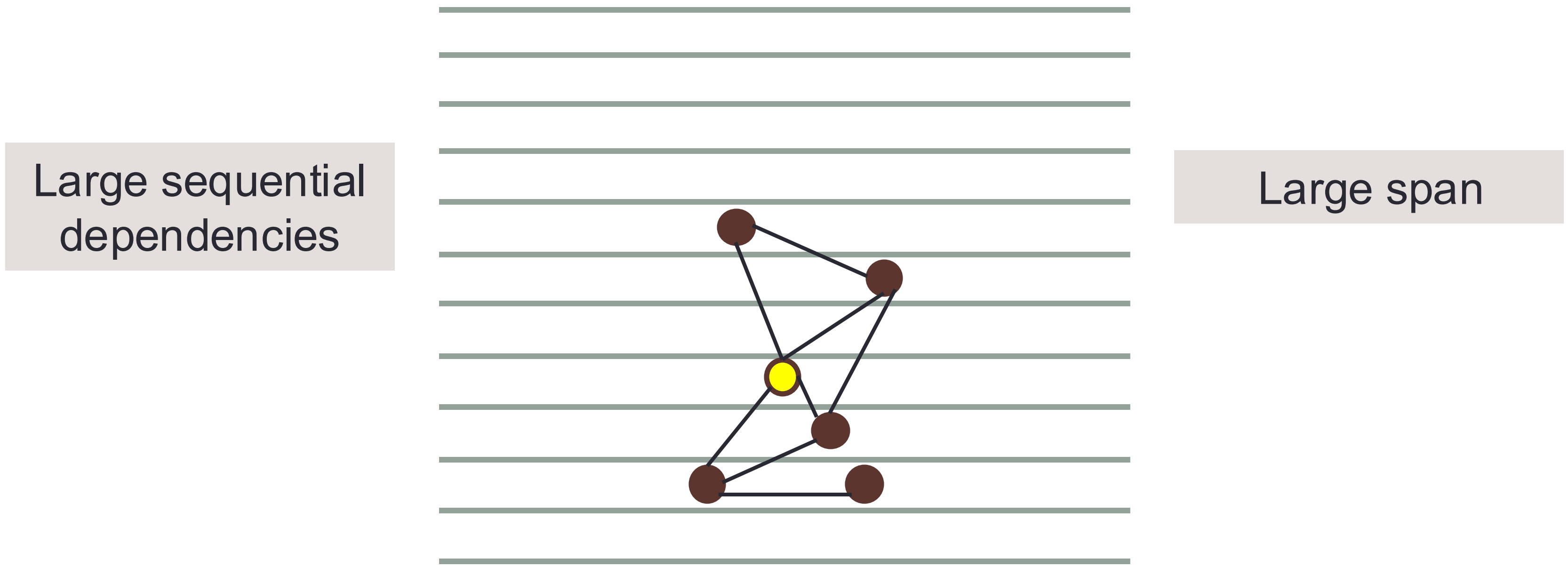

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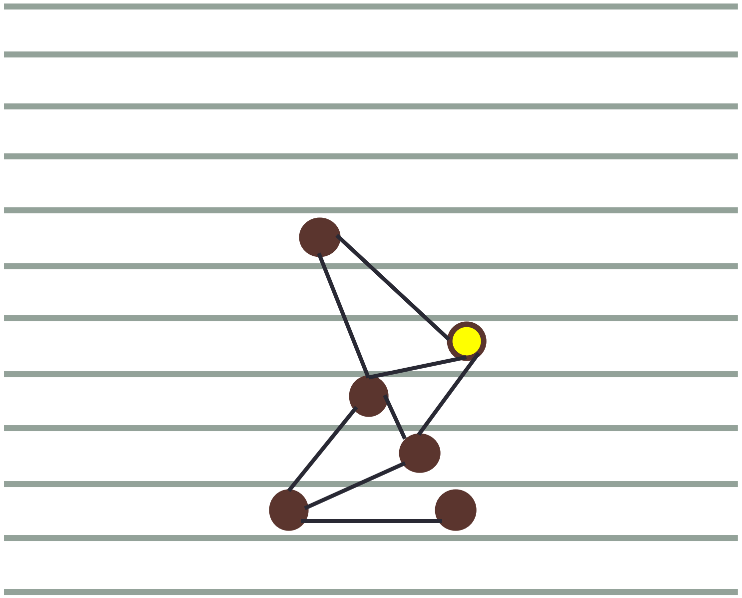
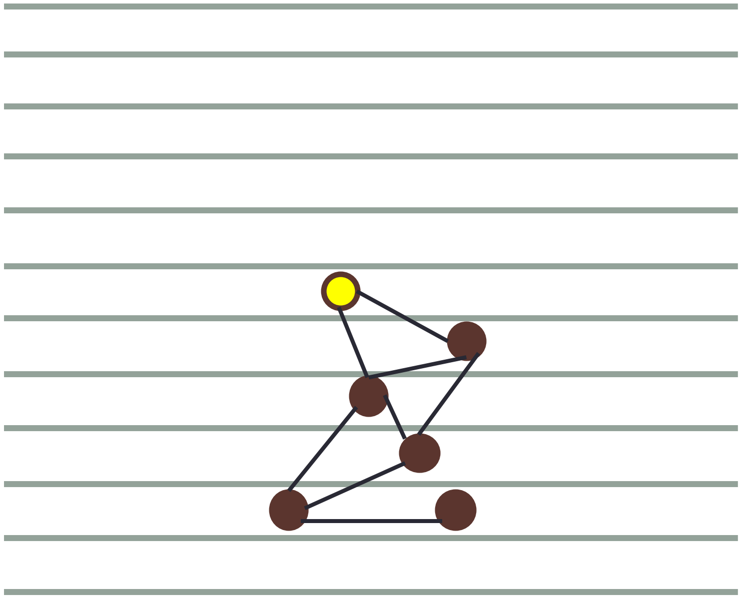


Large span

Difficulties with Parallelization

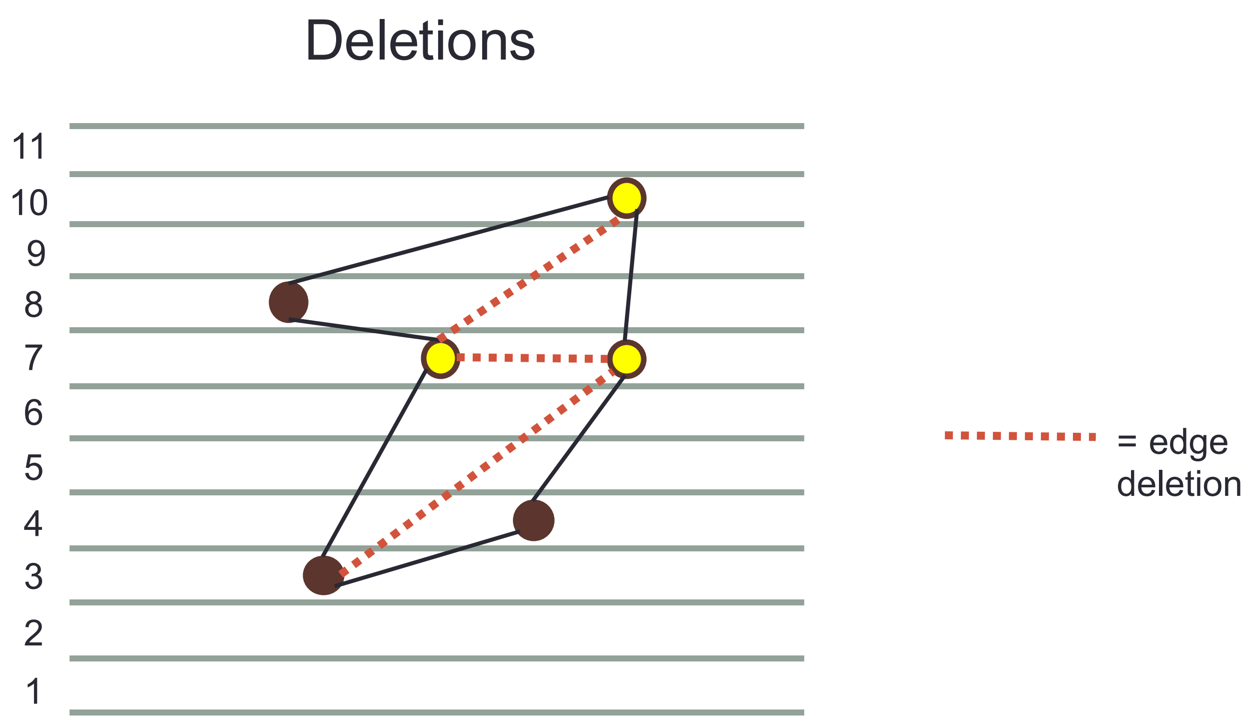
Large sequential dependencies

Only processes one update at a time



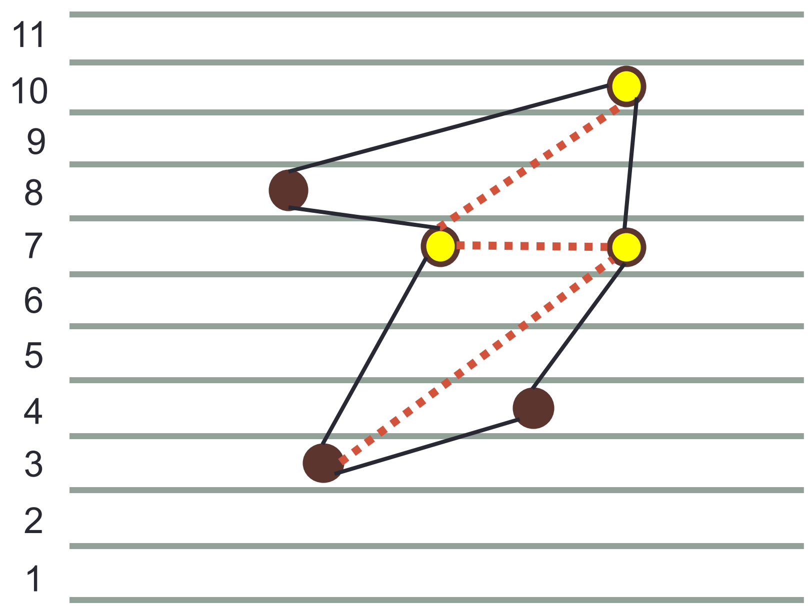
Large span

Our Parallel Batch-Dynamic Level Data Structure (PLDS)



Our Parallel Batch-Dynamic Level Data Structure (PLDS)

Deletions



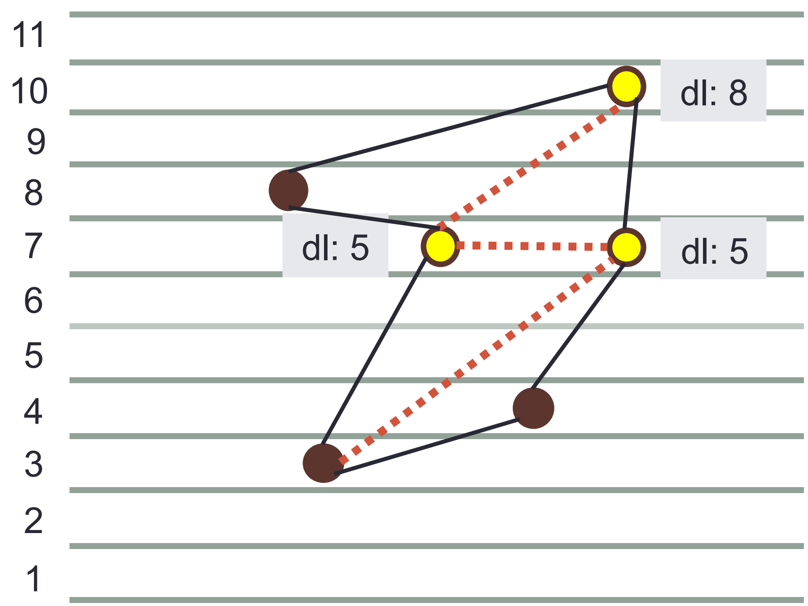
Only the lower bound invariant is ever violated.

- Vertices only need to move down, and never up

Our Parallel Batch-Dynamic Level Data Structure (PLDS)

Deletions

For vertices incident to updated edges, calculate *desire-level (dl)*: **closest level that satisfies invariants**



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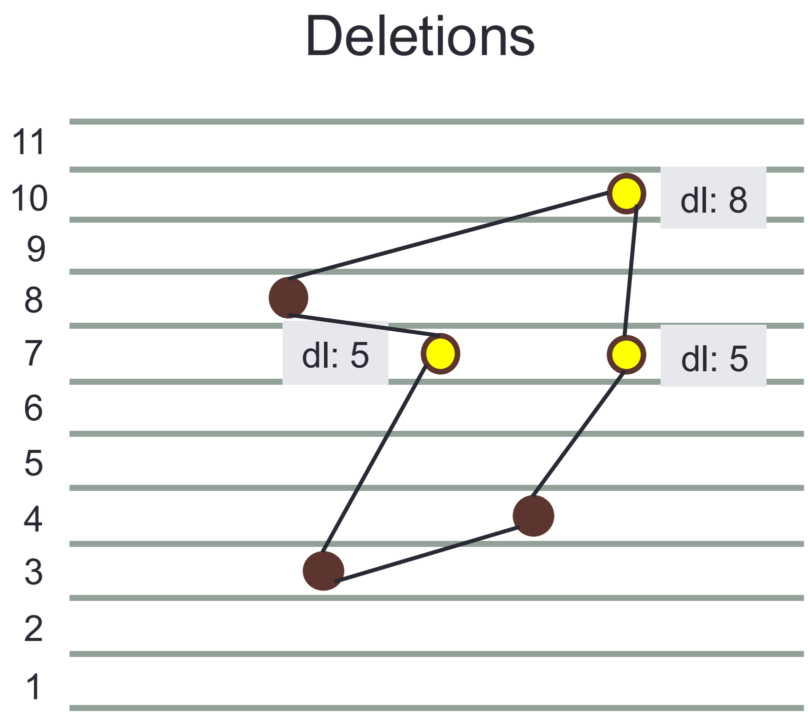

Iterate from **bottommost level to top level** and move vertices to desire-level

Only the lower bound invariant is ever violated.

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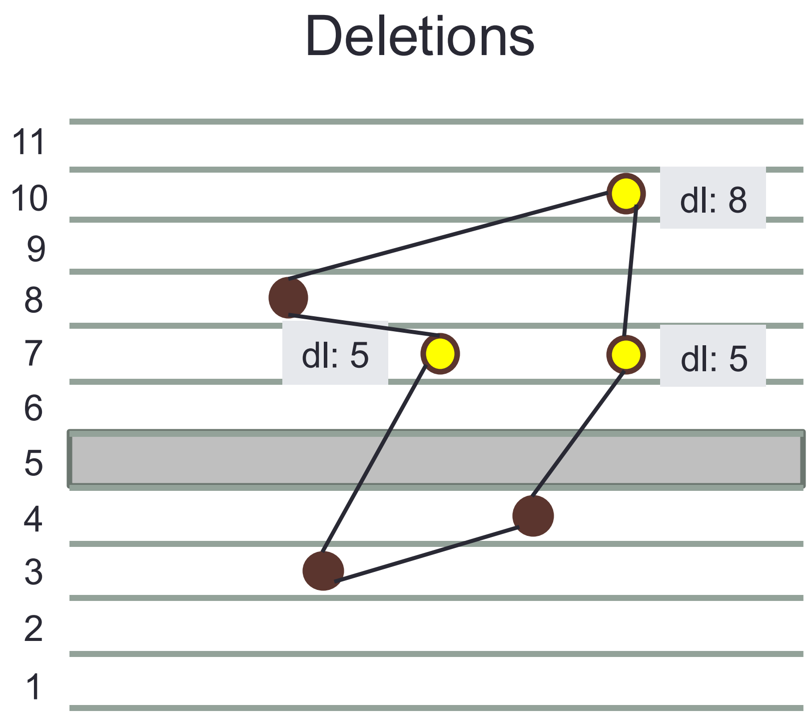

Iterate from **bottommost level to top level** and move vertices to desire-level

Only the lower bound invariant is ever violated.

To achieve parallelism (low span), we need to move all vertices together for each desire-level

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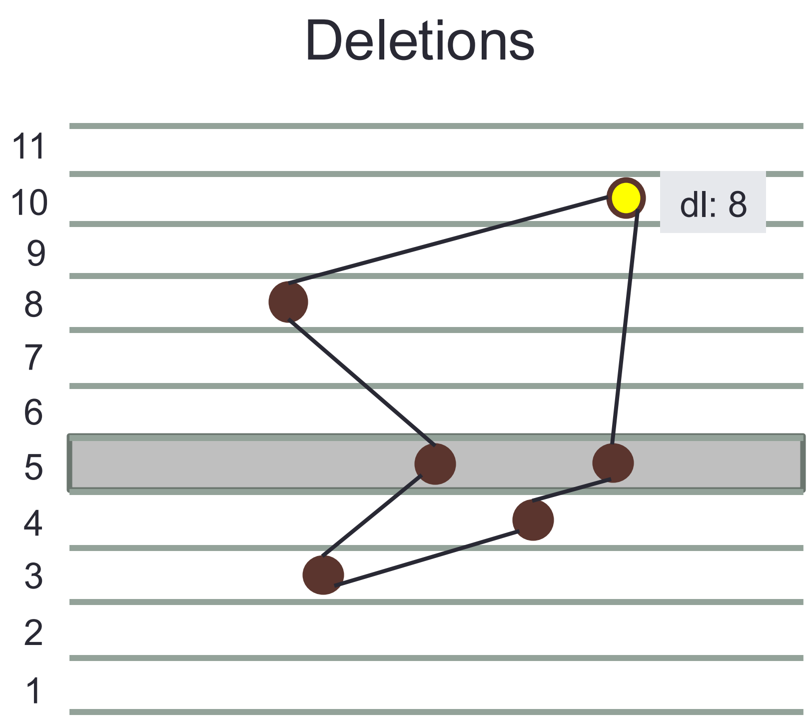

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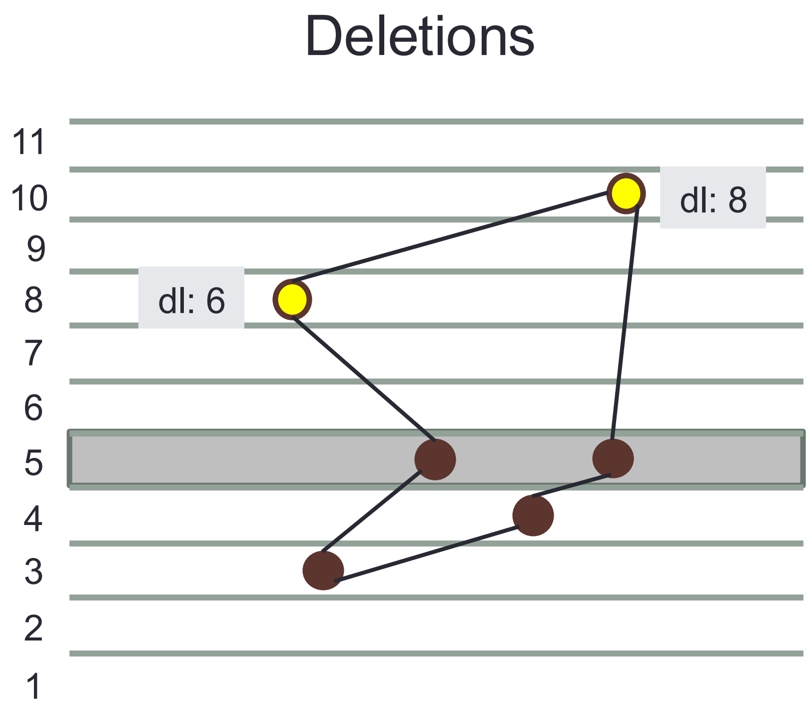

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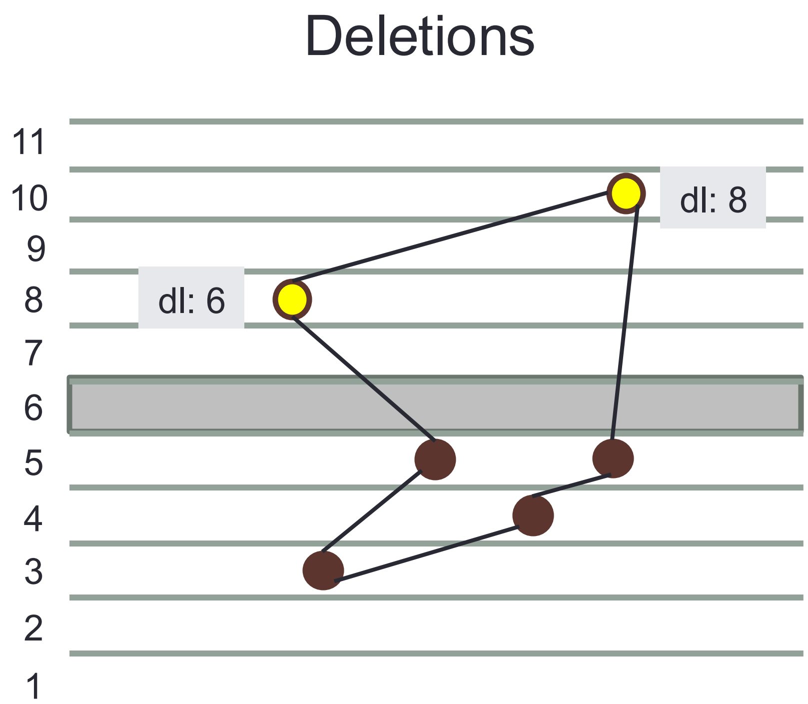

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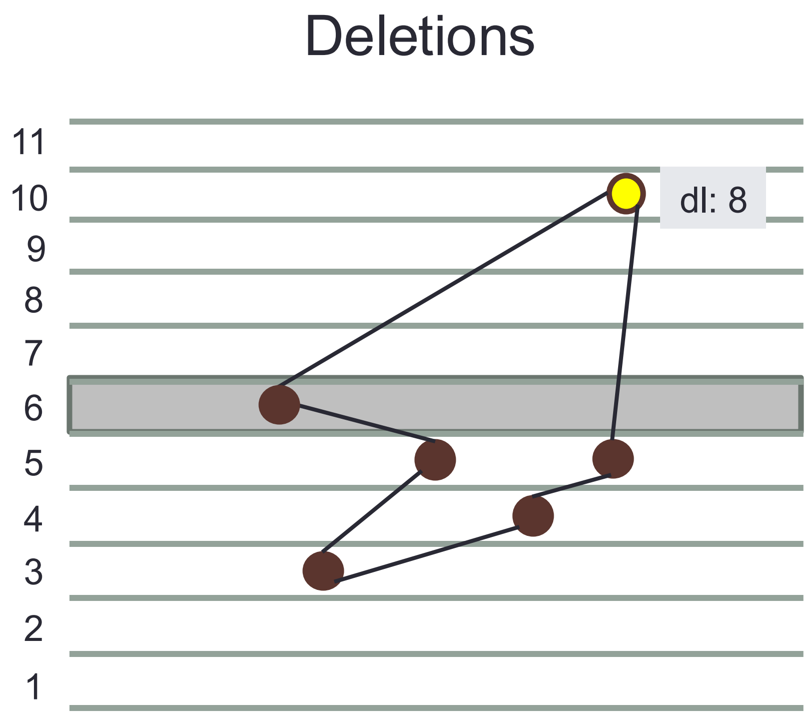

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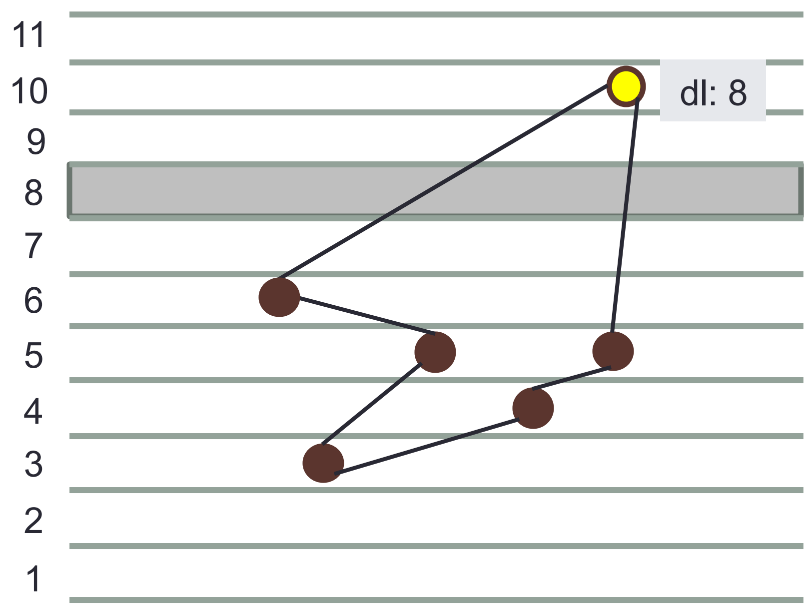
Deletions



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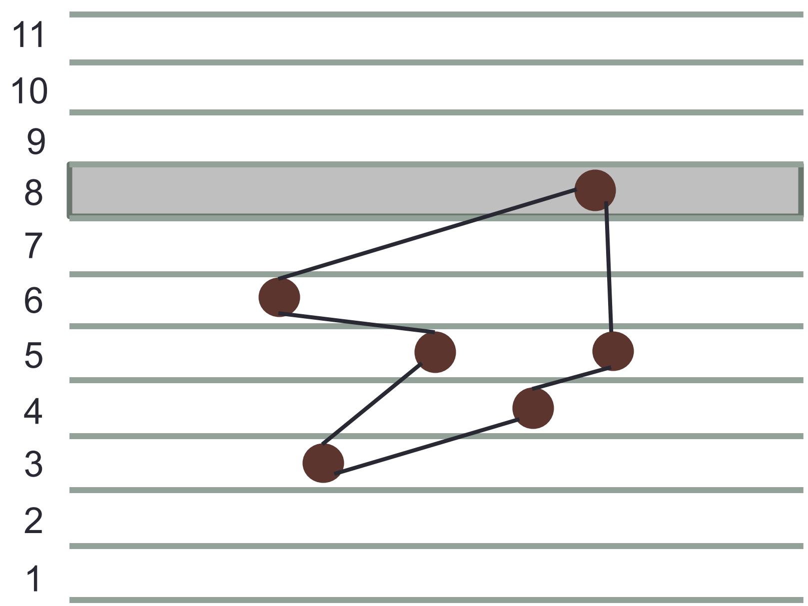
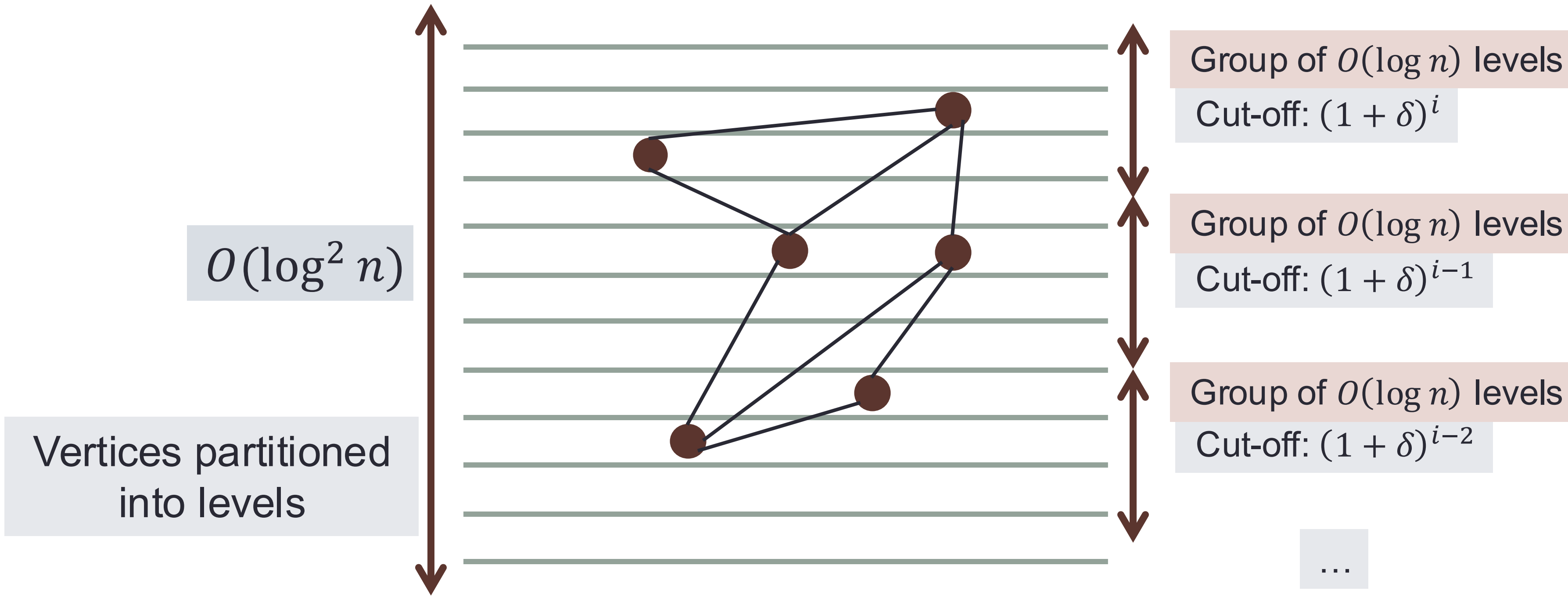


Iterate from **bottommost level to top level** and move vertices to desire-level

Only the lower bound invariant is ever violated.

- Each vertex moves only once, unlike in sequential LDS

Coreness Estimate



- We set the coreness estimate of a vertex to be $(1+\delta)^{\max(\lfloor (level(v)+1)/(4\lceil \log_{1+\delta} n \rceil) \rfloor - 1, 0)}$
- Exponent is roughly the group number
- Higher vertices have higher coreness estimates
- This gives a $(2+\epsilon)$-approximation
- Getting better than a 2-approximation is P-complete
- Automatically get $(4+\epsilon)$-approximation to densest subgraph value

Implementation Details

- Designed an optimized multicore implementation
- Used parallel primitives and data structures from the Graph Based Benchmark Suite [Dhulipala et al. ‘20]
- Maintain concurrent hash tables for each vertex v
  - One for storing neighbors on levels ≥ level(v)
  - One for storing neighbors on every level i in [0, level(v)-1]
- Moving vertices around in the PLDS requires carefully updating these hash tables for work-efficiency

Complexity Analysis

- $O(\log^2 n)$ levels
  - $O(\log \log n)$ span per level to calculate desire-levels using doubling search
  - $O(\log^* n)$ span with high probability for hash table operations
- Total span: $O(\log^2 n \log \log n)$

- $O(B \log^2 n)$ amortized work is based on potential argument
  - Uses very similar analysis to Bhattacharya, Henzinger, Nanongkai, Tsourakakis [2015]
  - Vertices and edges store potential based on their levels in PLDS, which is used to pay for the cost of moving vertices around
  - We need to map parallel operations to an equivalent set of sequential operations

Experiments

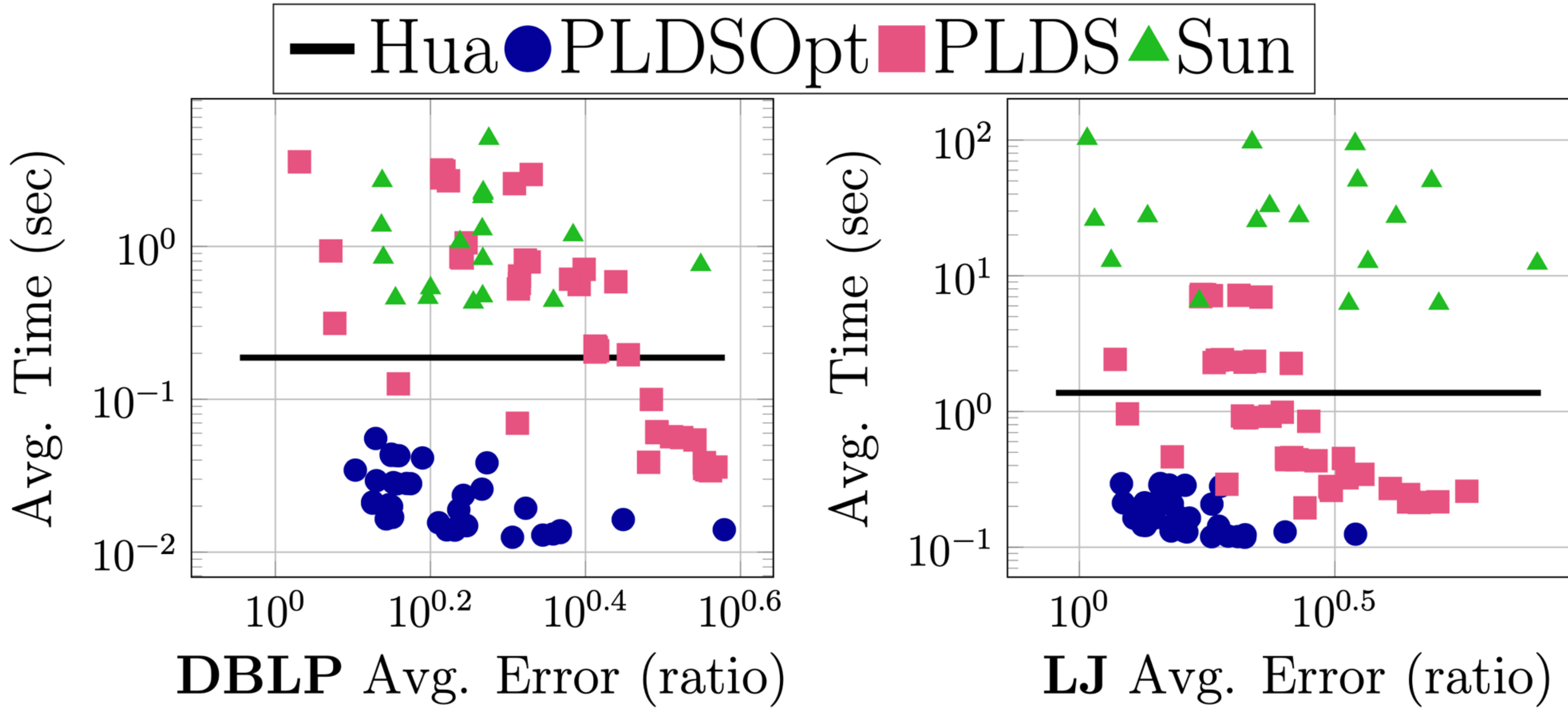
Experimental Setup

- c2-standard-60 Google Cloud instances
  - 30 cores with two-way hyper-threading
  - 236 GB memory
- m1-megamem-96 Google Cloud instances
  - 48 cores with two-way hyperthreading
  - 1433.6 GB memory
- 3 different types of batches:
  - All batches of insertions
  - All batches of deletions
  - Mixed batches of both insertions and deletions

Runtimes/Accuracy vs. State-of-the-Art Algorithms

PLDS: our algorithm
PLDSOpt: optimized PLDS

Hua et al.: **parallel, exact, dynamic** algorithm
Sun et al.: **sequential, approx., dynamic** algorithm



PLDSOpt: 19–544x speedup over Sun et al.

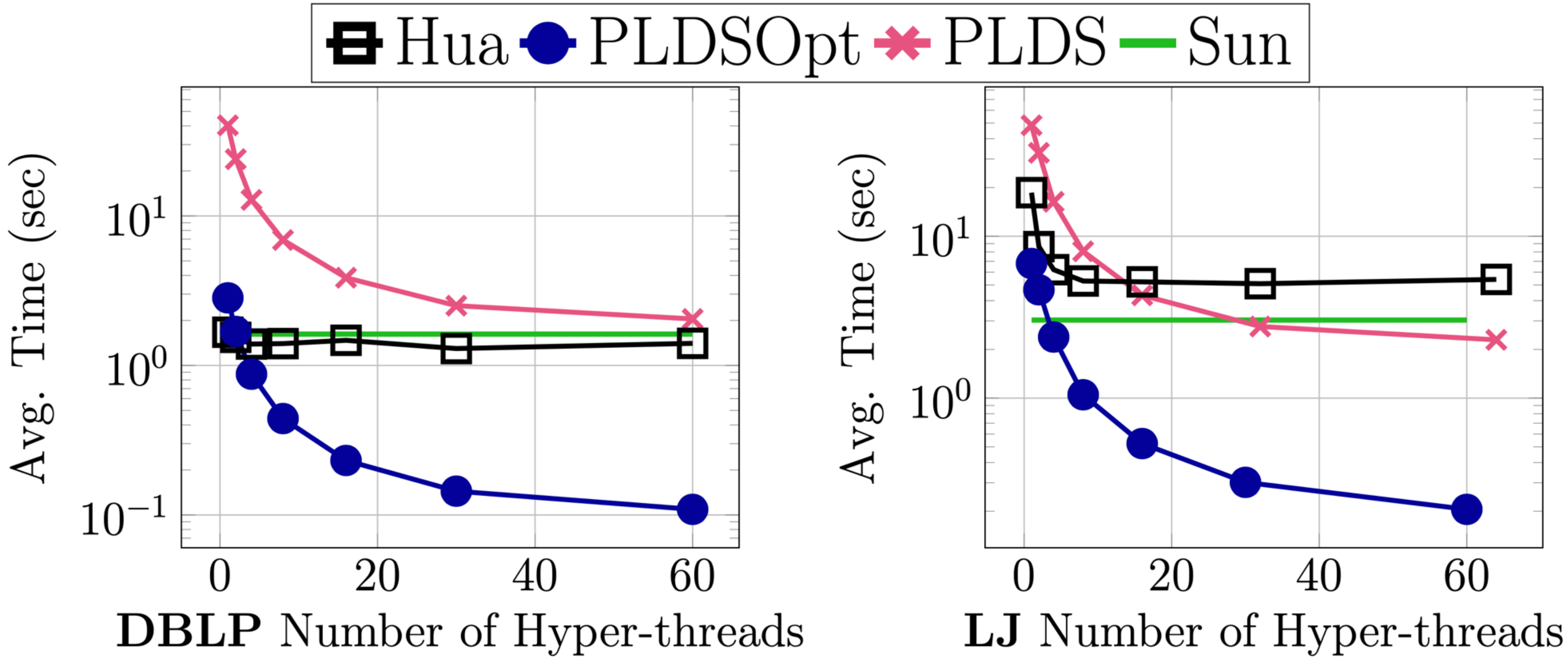
ces, 2.1M edges
4.8M vertices, 85

PLDSOpt: 2.5–25x speedup over Hua et al.

Scalability vs. # Hyper-threads

PLDS: our algorithm
PLDSOpt: optimized PLDS

Hua et al.: **parallel, exact, dynamic** algorithm
Sun et al.: **sequential, approx., dynamic** algorithm

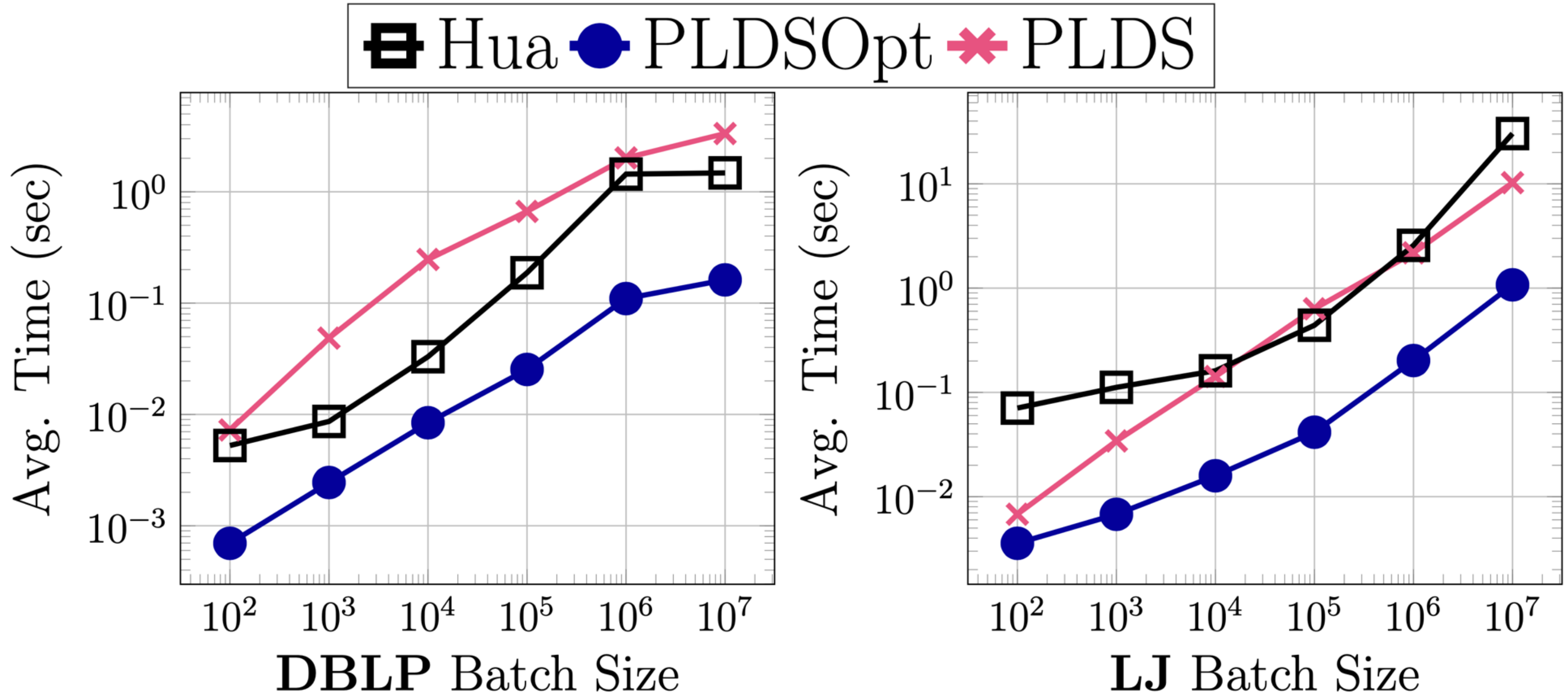


- Self-relative parallel speedups
  - PLDSOpt: 33x, PLDS: 26x, Hua: 3.6x
- PLDSOpt is faster than all of the other algorithms at 4 or more cores

Runtime vs. Batch Size

PLDS: our algorithm
PLDSOpt: optimized PLDS

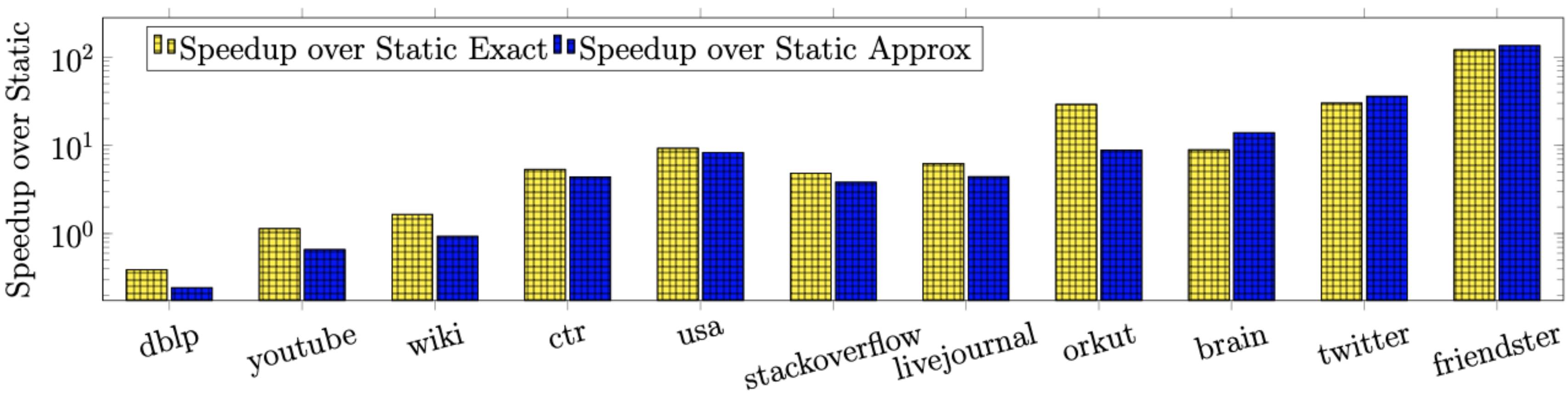
Hua et al.: **parallel, exact, dynamic** algorithm
(**Sun et al.** does not have a batch method)



- PLDSOpt achieves 2.5-115x speedup over Hua et al.

Runtime vs. Static Algorithms

- Parallel exact *k*-core decomposition [Dhulipala, Blelloch, Shun 2018]
- Parallel $(2 + \epsilon)$-approximate *k*-core decomposition



Batch size = 10^6 Graphs ordered by size (left to right)

- We achieve speedups for all but the smallest graphs
- Speedups of up to 122x for Twitter (1.2B edges) and Friendster (1.8B edges)

Conclusion

- Theoretically-efficient and practical batch-dynamic *k*-core decomposition algorithm
- Using our PLDS, we designed parallel batch-dynamic algorithms for several other problems:
  - Low out-degree orientation
  - Maximal matching
  - Clique counting
  - Graph coloring
- Source code available at https://github.com/qqliu/batch-dynamic-kcore-decomposition

Quanquan C. Liu, Jessica Shi, Shangdi Yu, Laxman Dhulipala, Julian Shun, *"Parallel Batch-Dynamic Algorithms for k-Core Decomposition and Related Graph Problems,"* SPAA 2022