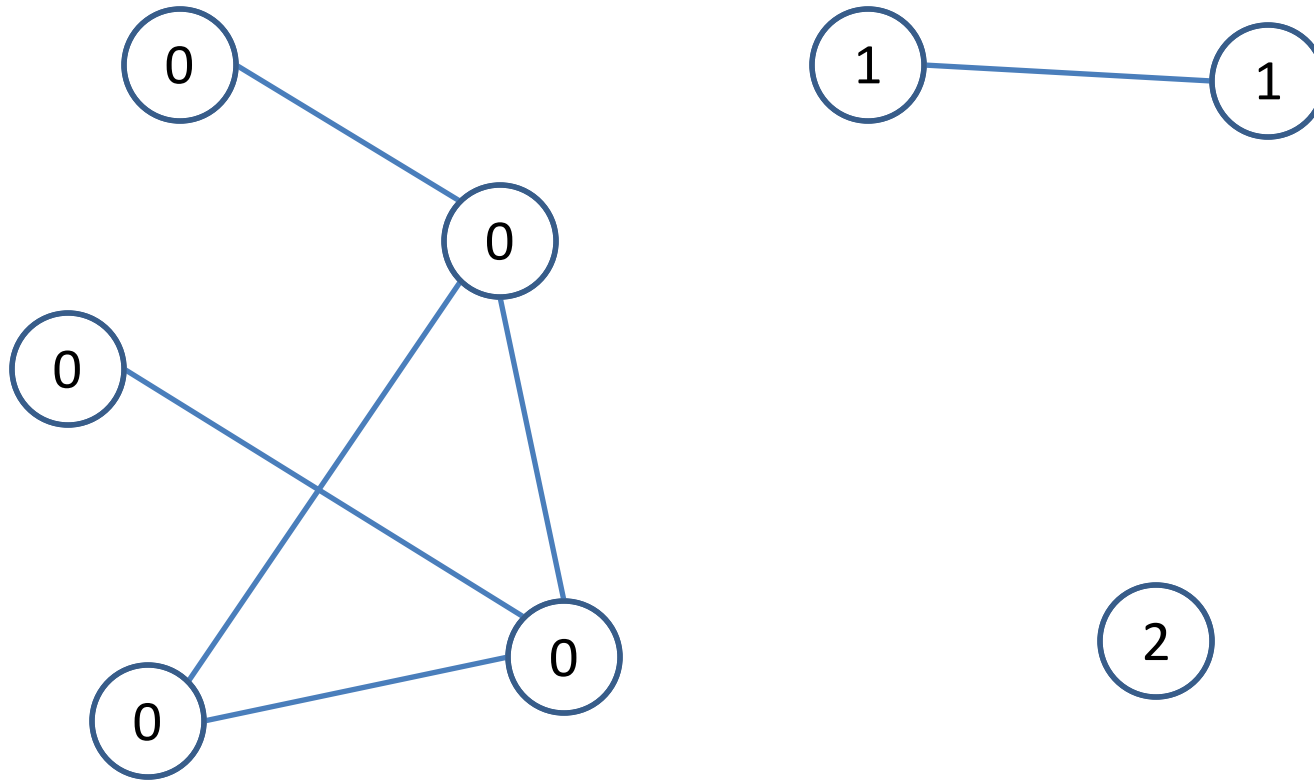


A Simple and Practical Linear-Work Parallel Algorithm for Connectivity

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Connected Component Labeling



Given an undirected graph, label all vertices such that $L(u) = L(v)$ if and only if there is a path between u and v

Connected Component Labeling

- What are some simple algorithms?
 - Depth-first search
 - Linear work/span
 - Versions of DFS that are parallel are not work-efficient
 - Breadth-first search
 - Linear work
 - Parallelism limited by graph diameter
 - Polylogarithmic span version not work-efficient
 - Spanning forest
 - Good parallelism
 - Practical parallel implementations not linear work

Connected Component Labeling

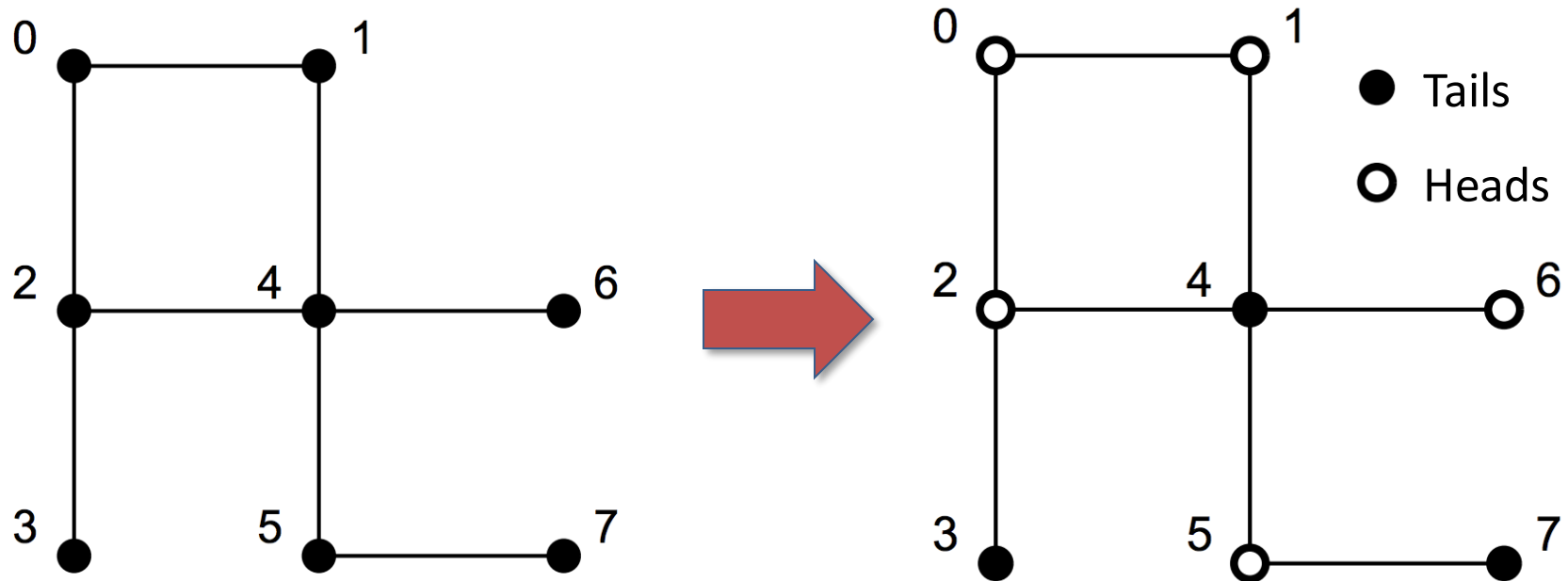
- Parallel (polylogarithmic span) algorithms
 - Shiloach and Vishkin, Awerbuck and Shiloach
 - Combines (contracts) vertices in each iteration
 - $O(m \log n)$ work, $O(\log n)$ span
 - Reif, Phillips
 - Uses randomization to simplify contraction algorithms
 - $O(m \log n)$ work and $O(\log n)$ span w.h.p.
 - $O(\log n)$ rounds but don't guarantee a constant fraction of edges removed
 - $O(m)$ work algorithms
 - Gazit '91, Halperin/Zwick '96, Cole et al. '96, Poon/Ramachandran '97, Pettie/Ramachandran '02
 - Quite complicated and no implementations exist

Our Contributions

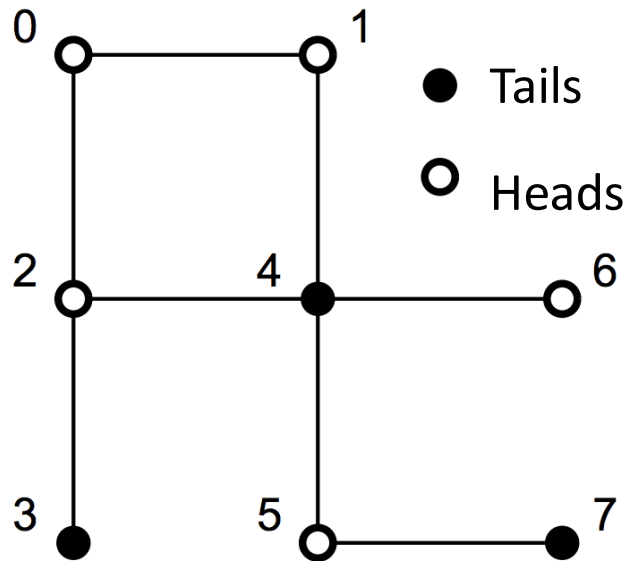
- Practical parallel connectivity algorithm with linear work and polylogarithmic span
- Experimental evaluation: competitive with existing parallel implementations (that are not linear-work and polylogarithmic span)

Previous Work: Random Mate

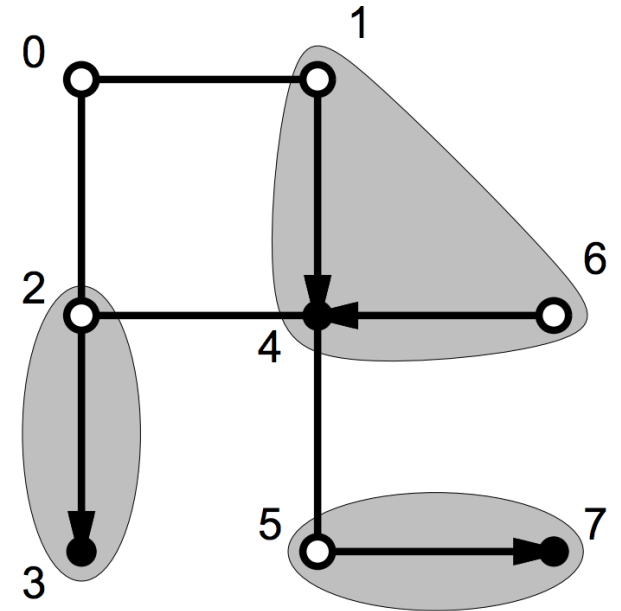
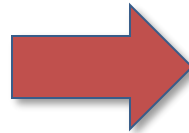
- Idea: Form a set of non-overlapping star subgraphs and contract them
- Each vertex flips a coin. For each Heads vertex, pick an arbitrary Tails neighbor (if there is one) and point to it



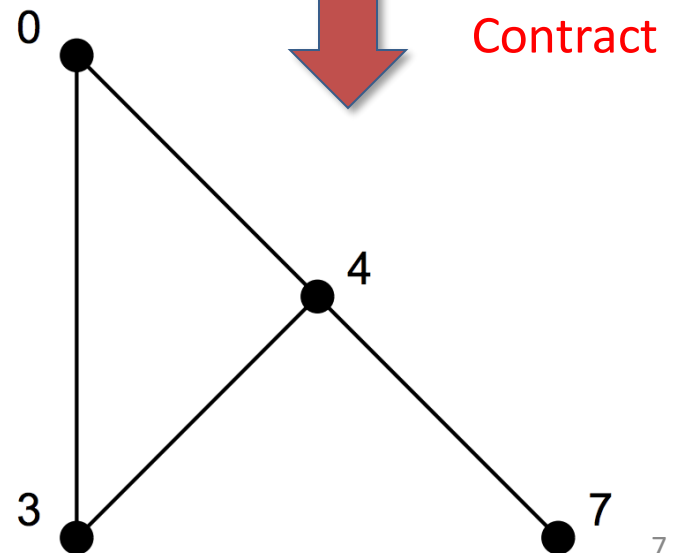
Previous Work : Random Mate



Form stars



Contract



Repeat until each component has a single vertex

Expand vertices back in reverse order with label of neighbor

Previous Work : Random Mate

CC_Random_Mate(L, E)

if(|E| = 0) Return L *//base case*

else

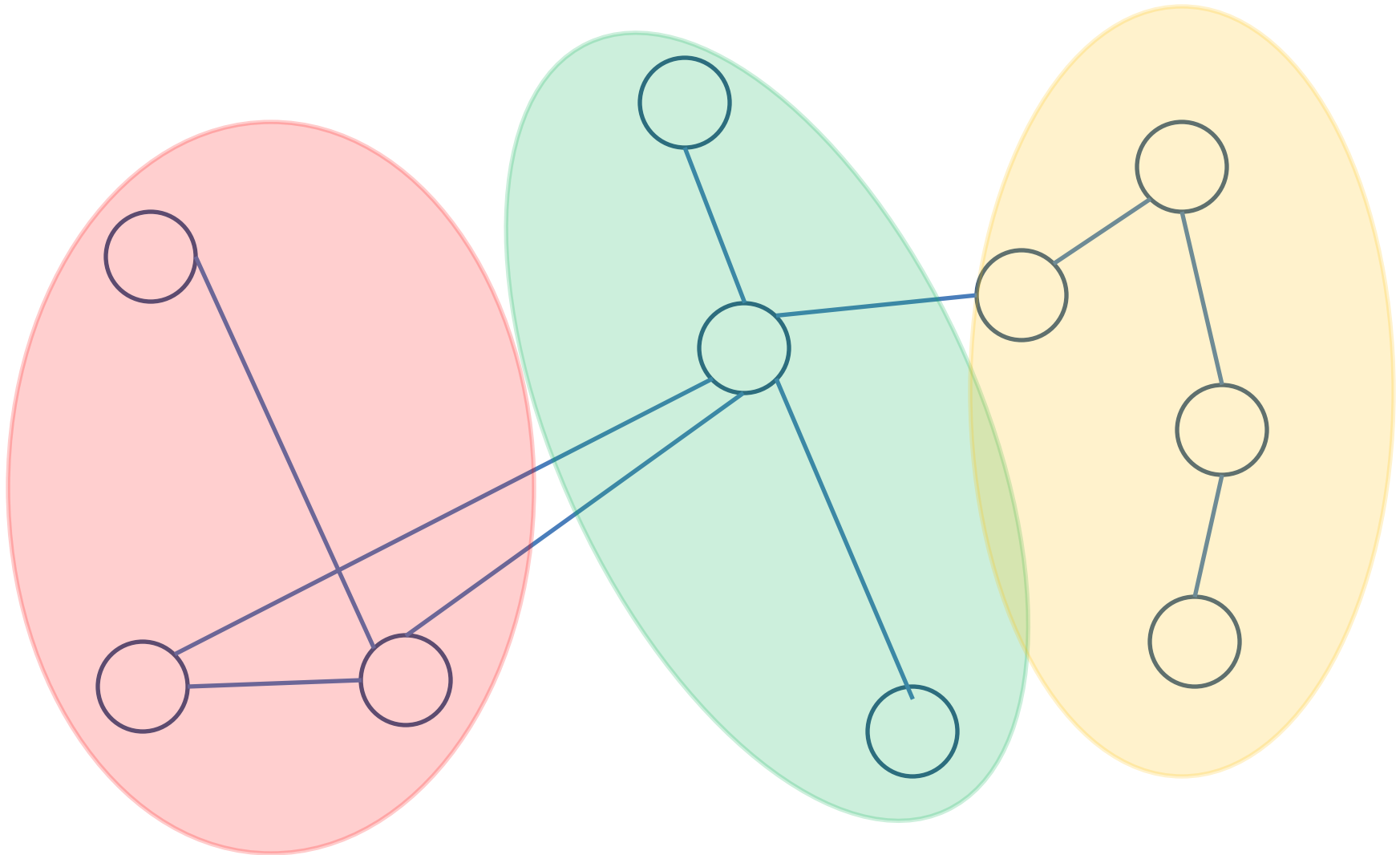
1. Flip coins for all vertices
2. For v where $\text{coin}(v)=\text{Heads}$, hook to arbitrary Tails neighbor w and set $L(v) = L(w)$
3. $E' = \{ (L(u), L(v)) \mid (u, v) \in E \text{ and } L(u) \neq L(v) \}$
4. $L' = \text{CC_Random_Mate}(L, E')$
5. For v where $\text{coin}(v)=\text{Heads}$, set $L'(v) = L'(w)$ where w is the Tails neighbor that v hooked to in Step 2
6. Return L'

- Each iteration requires $O(m+n)$ work and $O(1)$ span
 - Assumes we do not pack vertices and edges
- Each iteration eliminates at least $1/4$ of the vertices in expectation $\rightarrow O(\log n)$ rounds w.h.p.

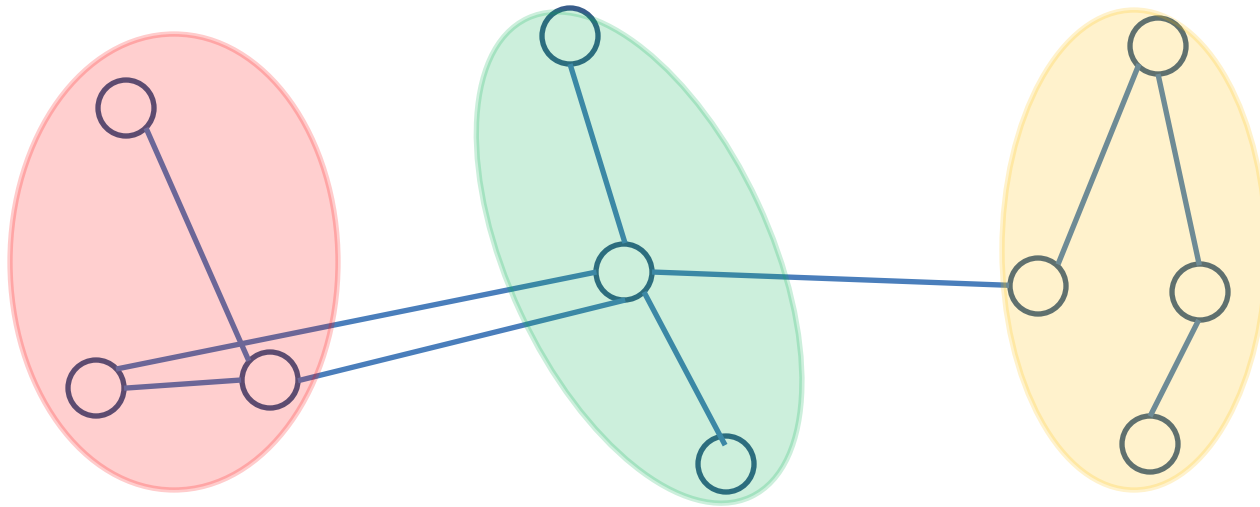
$$W = O(m \log n) \text{ w.h.p.}$$

$$S = O(\log n) \text{ w.h.p.}$$

Low diameter decomposition



Low diameter decomposition

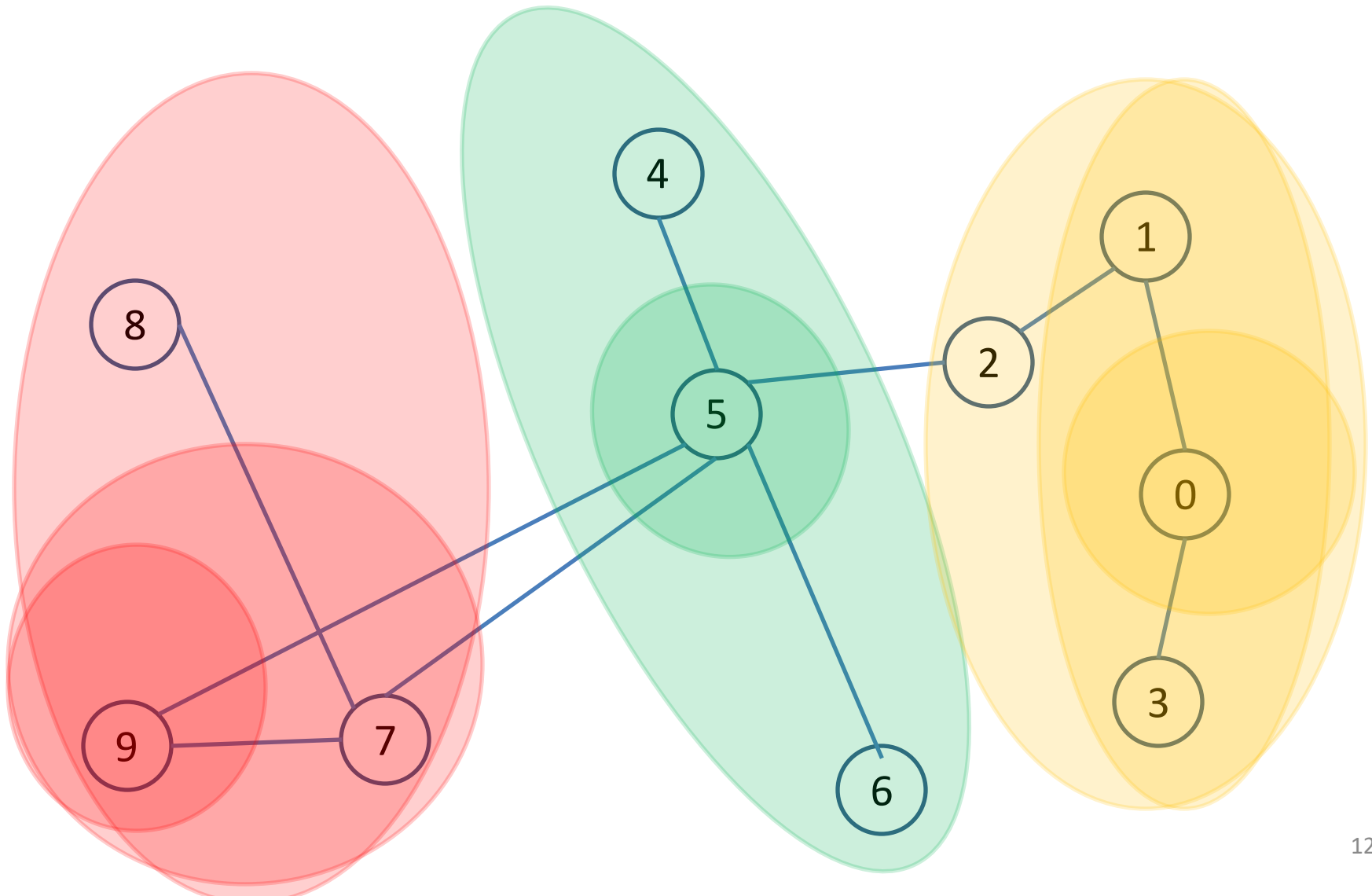


- (β, d) -decomposition ($0 < \beta < 1$) partitions V into V_1, \dots, V_k such that
 - The shortest path between any two vertices in a partition is at most d
 - The number of inter-partition edges is at most βm
- Used in linear system solvers and metric embeddings

Low diameter decomposition

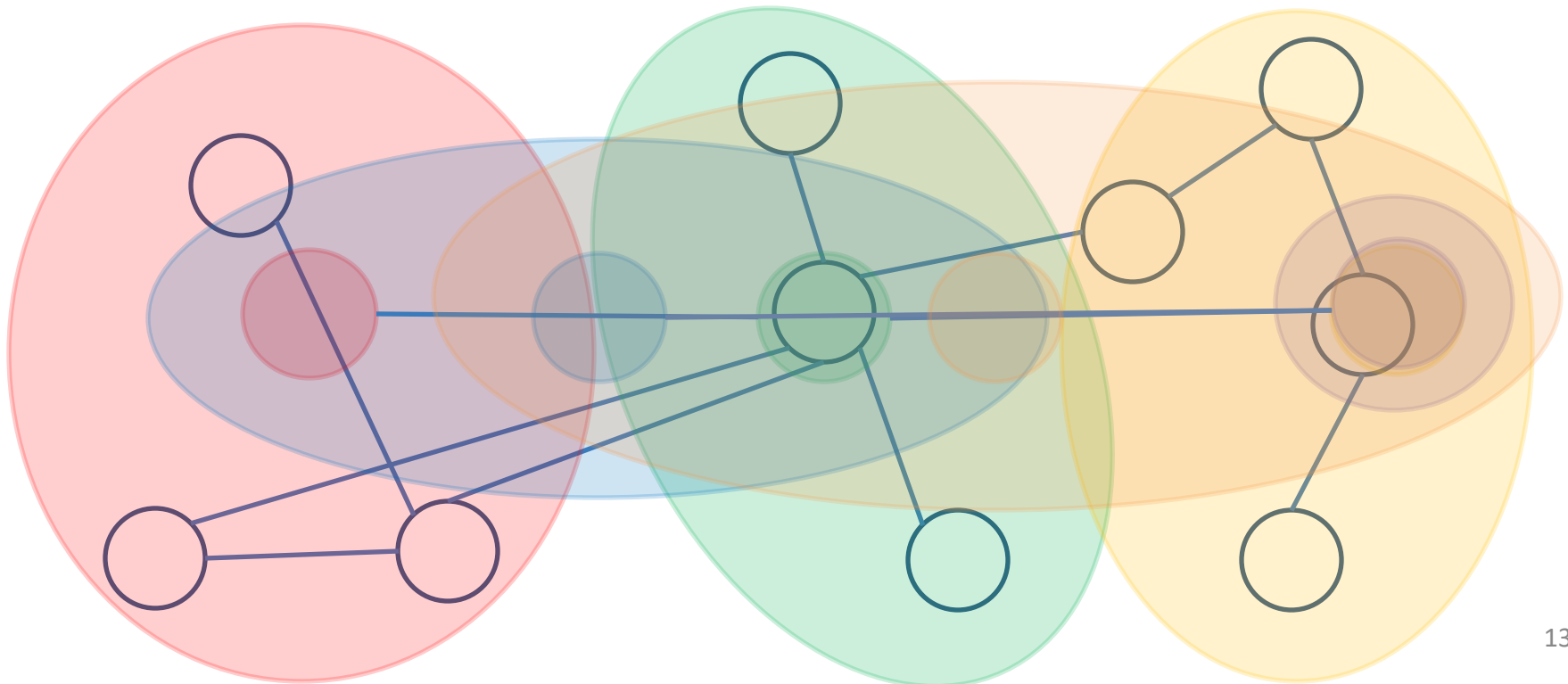
- A $(\beta, O(\log n / \beta))$ -decomposition can be computed in $O(m)$ expected work and $O(\log^2 n / \beta)$ span w.h.p. [Miller et al. 2013]
 - Start breadth-first searches from vertices with exponentially-distributed (parameter β) start times
 - Each BFS creates a partition containing the source and all vertices explored
 - A BFS does not explore vertices already visited by another BFS
 - All vertices will have started BFS or been explored by time $O(\log n / \beta)$
 - BFS's are work-efficient and terminate in $O(\log n / \beta)$ iterations.
 - Each iteration requires $O(\log n)$ span.
 - Bounding number of inter-partition edges:
 - An edge is cut with probability at most β due to properties of exponential distribution
 - Linearity of expectations gives at most βm edges cut

Low diameter decomposition example



Our Connectivity Algorithm

- Compute a $(\beta, O(\log n / \beta))$ -decomposition
- Contract each partition into a single vertex
- Recurse



Our Connectivity Algorithm

- Compute a $(\beta, O(\log n / \beta))$ -decomposition
- Contract each partition into a single vertex
- Recurse

Analysis for $\beta=1/2$

- Assume contraction can be done in linear work and in $O(\log n)$ span
- $m/2$ edges remain after each round in expectation
 - Work = $O(m) + O(m/2) + \dots = O(m)$ in expectation
- $O(\log n)$ levels of recursion suffice w.h.p.
 - Span = $O(\log n) * O(\log^2 n / \beta) = O(\log^3 n)$ w.h.p.

Contraction

- Contraction can be done in $O(\log n)$ span with bookkeeping and parallel prefix sums
 - Intra-partition edges are filtered out in $O(m)$ work and $O(\log n)$ span
 - Prefix sums: relabel vertices to smaller range
 - Duplicate edges removed using parallel hashing in $O(m)$ work and $O(\log n)$ span
 - Not needed theoretically

Improving span

- Each round of BFS can be implemented in $O(\log^* n)$ span w.h.p. using approximate prefix sum and compaction [Gil-Matias-Vishkin '91, Goodrich-Matias-Vishkin '94]
 - Improves span of low diameter decomposition to $O(\log n \log^* n)$
- Recurse for $O(\log \log n)$ rounds
 - Left with $O(m/\log n)$ edges
 - Switch to $O(m \log n)$ work, $O(\log n)$ span algorithm
- Result: Linear work algorithm with $O(\log n \log \log n \log^* n)$ span w.h.p.

Low diameter decomposition variants

- Resolving conflicts among BFS's
 - Decomp-min: breaks ties deterministically
 - Miller et al. showed this produces $(\beta, O(\log n/\beta))$ -decomposition
 - Uses write-with-min (via compare-and-swap)
 - Requires two phases
 - Decomp-arb: breaks ties arbitrarily
 - We prove $(2\beta, O(\log n/\beta))$ -decomposition
 - Uses compare-and-swap
 - Requires just a single phase
 - Decomp-arb-hybrid: uses direction-optimizing BFS
 - This is the fastest one and used in the following experimental results

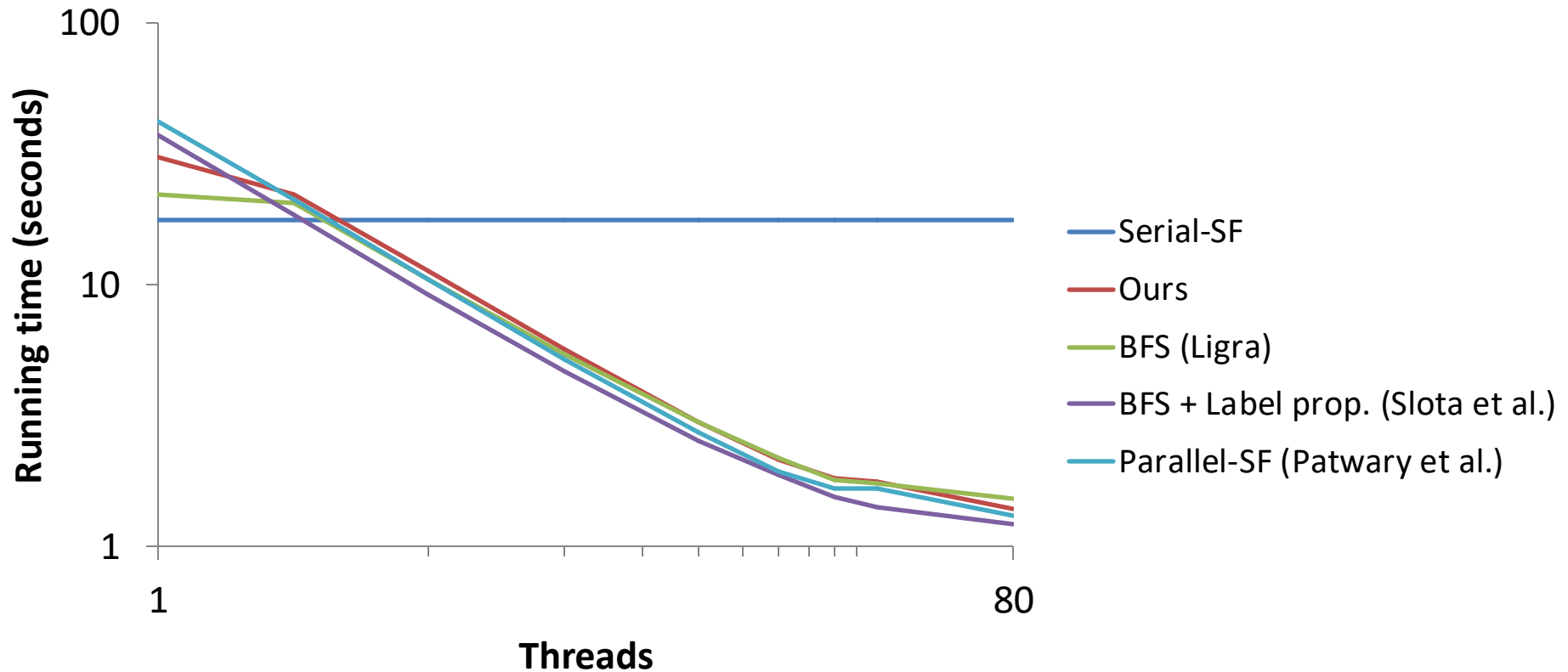
Experiments

- 40-core (with 2-way hyper-threading) Intel Nehalem machine
- Implemented in Cilk Plus
- 3 different implementations, but only showing best one
- Real-world and artificial graphs

Compare to existing implementations

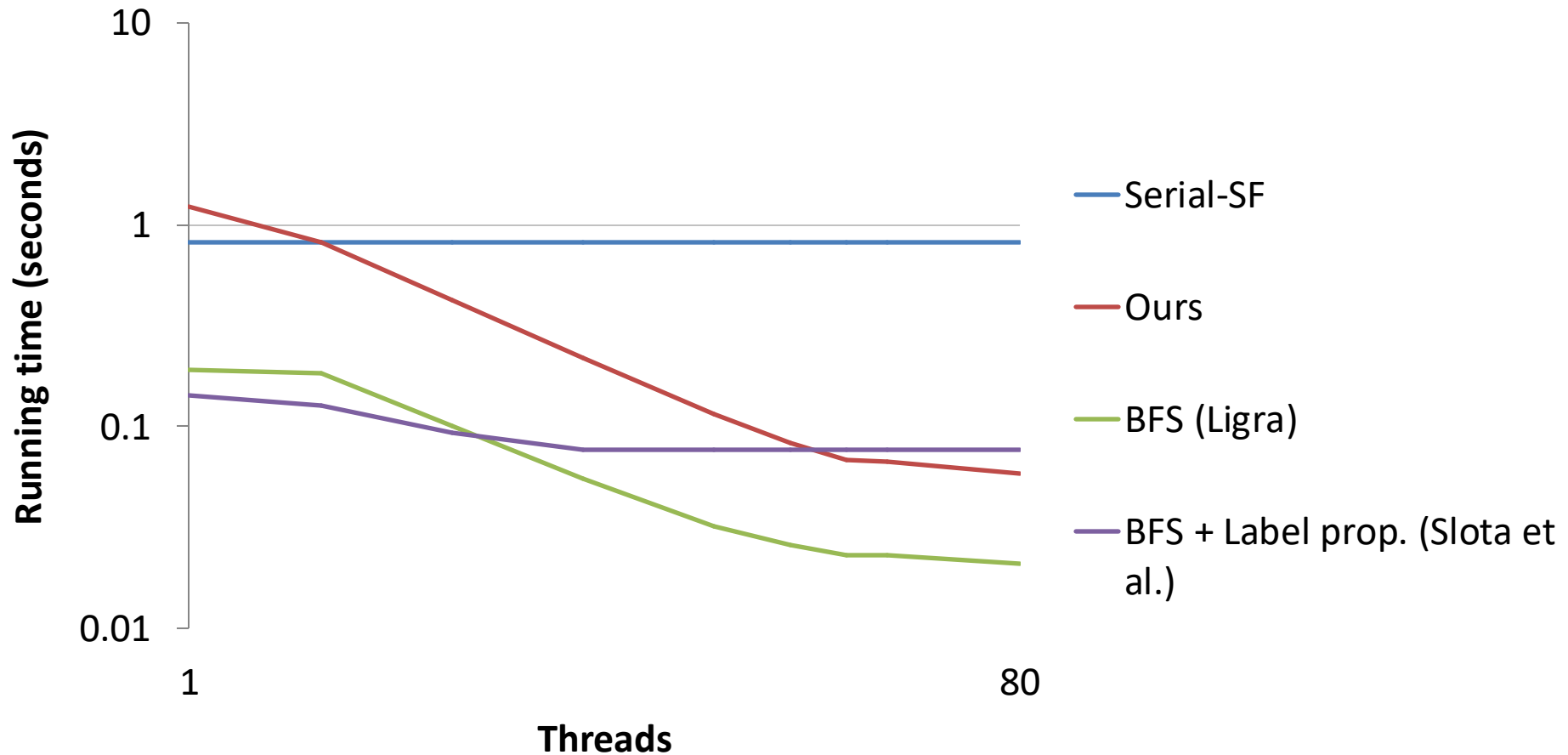
- Existing implementations
 - Sequential spanning forest
 - Parallel spanning forest (Problem Based Benchmark Suite)
 - Parallel spanning forest (Patwary et al.)
 - Parallel BFS (Ligra)
 - Parallel BFS + Label propagation (Slota et al.)
- None provably linear work and polylog span

3D grid graph ($n = 10^8$, $m = 3 \times 10^8$)



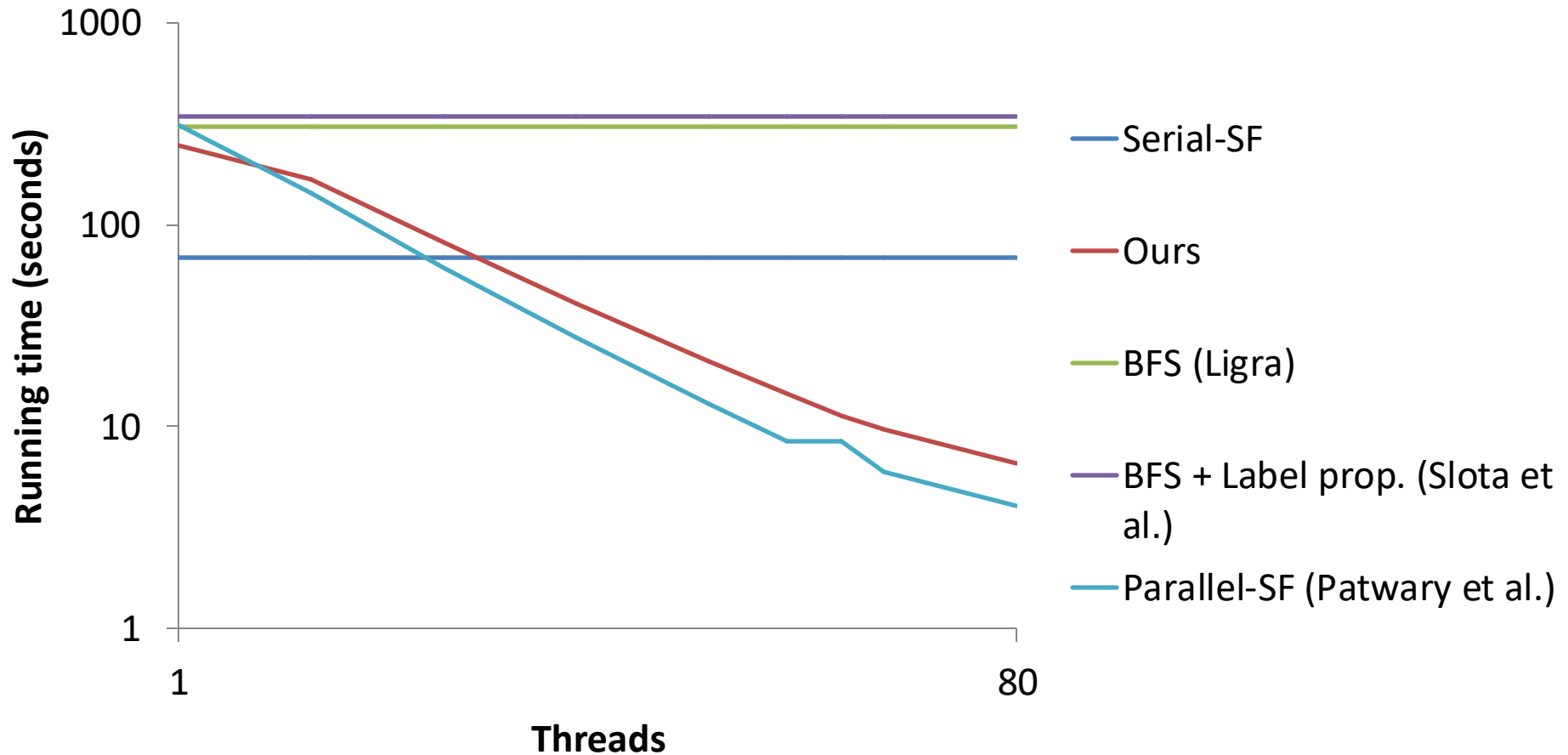
- Competitive with other implementations

com-Orkut graph ($n \approx 3 \times 10^6$, $m \approx 10^8$)



- Fastest implementation uses single BFS

Line graph ($n = 5 \times 10^8$, $m = 5 \times 10^8$)



- Algorithms based on single BFS do poorly

Our algorithm is competitive

- No “worst-case” inputs
- Performance always close to the fastest implementation for any graph
 - Only at most 70% slower than spanning forest algorithms, and usually much less
 - Can be faster or slower than BFS, depending on graph diameter
- Up to 13x speedup on 40 cores relative to sequential
- 18—39x self-relative speedup

Conclusion

- Simple and practical linear-work, polylog-span connectivity algorithm
 - Can be easily modified to compute spanning forest
- As far as we know, first to be both practical and have linear work and polylog span
- Implementations competitive with existing parallel implementations
- Future direction: Can similar ideas give us a practical linear-work parallel algorithm for minimum spanning forest?