

Parallel Graph Decompositions Using Random Shifts

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Graph Decompositions

On graph $G = (V, E)$, the strong and weak diameter of $S \subseteq V$ are the maximum shortest-path distances between any two vertices in S , with paths:

- ▶ **strong diameter**: using only vertices in S .
- ▶ **weak diameter**: allowing shortcuts through $V \setminus S$.

Definition

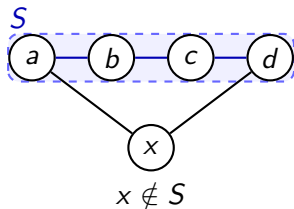
A (β, d) decomposition partitions V into pieces $S_1, S_2, \dots, S_k$ where each S_i has (strong) diameter at most d , and the number of edges with endpoints in different pieces is at most βm .

Background

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- ▶ Over arbitrary graphs, the best value of d is $\Theta(\log n / \beta)$.
- ▶ Useful for e.g. low-stretch spanning trees.
 - ▶ Take a decomposition and BFS on each cluster.
 - ▶ Contract all the clusters.
 - ▶ Recursively repeat on the smaller graph.



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► **“Ball carving” algorithm:**

- Pick arbitrary vertex $u \in V$ and let $S = \{u\}$.
 - Repeatedly add all neighbors of S into S via BFS.
 - Stop when the number of edges in the cut of S is less than β of the edges in S .
 - Remove S from the graph and repeat.
- Work $O(m)$, but can't be parallelized.

Prior Work

Blelloch, Gupta, Koutis, Miller, Peng, Tangwongsan, SPAA 2011.

- ▶ Works in stages, each stage containing increasing density of centers.
- ▶ Do ball carving on each center, with uniformly random delay.
- ▶ Expected work $O(m \log^2 n)$, span $O(\log^3 n / \beta)$. Yields a $(\beta, O(\log^4 n / \beta))$ decomposition.

Current algorithm achieves $(\beta, O(\log n / \beta))$ decomposition with work $O(m)$ and span $O(\log^2 n / \beta)$.

Algorithm

Given undirected unweighted graph $G = (V, E)$ and parameter β :

- ▶ Each vertex $u \in V$ picks random shift δ_u independently from exponential distribution with mean $1/\beta$.
- ▶ Each vertex v is assigned to cluster S_u for vertex u which minimizes $\text{dist}_{-\delta}(u, v) = \text{dist}(u, v) - \delta_u$.
- ▶ **In parallel setting:** find the $\delta_{\max} = \max\{\delta_u \mid u \in V\}$.
- ▶ At each vertex u , start a BFS at time $\delta_{\max} - \delta_u$. A vertex v joins the first BFS that reaches it.

The decomposition has expected cut size $O(\beta m)$ and strong diameter $O(\log n/\beta)$ whp.

Proof Idea

- ▶ The strong diameter is bounded by $2\delta_{\max}$.
- ▶ For each edge $e = (x, y) \in E$, let z be the midpoint. If the difference between the first and second arrival times to z is at least 1, e is in a single cluster.

Recall the exponential distribution with mean $1/\beta$:

- ▶ PDF: $f(x) = \beta \exp(-\beta x)$ for $x \geq 0$.
- ▶ CDF: $F(x) = 1 - \exp(-\beta x)$.
- ▶ $\frac{\Pr[\delta > T+t]}{\Pr[\delta > T]} = \exp(-\beta t)$.

Strong Diameter

- ▶ If $v \in S_u$, any shortest path from u to v is contained in S_u .
- ▶ So strong diameter of S_u is at most (double) the distance from u to any $v \in S_u$.
- ▶ Bounded above by $2\delta_{\max}$: if $\text{dist}(u, v) > \delta_{\max}$, v arrives before u reaches v .
- ▶ We have

$$\Pr\left(\delta_{\max} > C \frac{\log n}{\beta}\right) \leq n \cdot \exp(-C \log n) = n^{-C+1}$$

Small Cut

- ▶ Fix edge $e \in E$ and midpoint z , and let $X_u = \delta_u - \text{dist}(u, z)$.
- ▶ Largest value of X_u arrives first. If it is at least 1 greater than the other X_u , both endpoints of e belong to S_u .
- ▶ For each $u \in V$, condition on δ_v for all $v \neq u$ and $X_u > M$ where $M = \max_{v \neq u}(X_v)$. Then

$$\Pr[X_u - M > 1] = \frac{\Pr[\delta_u > M + \text{dist}(u, z) + 1]}{\Pr[\delta_u > M + \text{dist}(u, z)]} \geq \exp(-\beta).$$

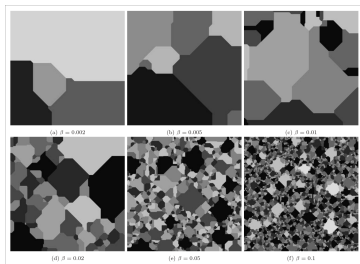
- ▶ Less than $1 - \exp(-\beta) \leq \beta$ that e is in the cut.
- ▶ Sum over edges to get βm expected cut size.

Complexity

- ▶ Algorithm has constant probability of success if we sample using rate $\beta/2$.
- ▶ Conditions can be checked in $O(m)$ work and $O(\log n)$ span.
- ▶ Parallel BFS checks each edge constantly many times, so expected work is $O(m)$.
- ▶ Each BFS layer takes $O(\log n)$ span, so expected span is $O(\log^2 n/\beta)$.

Opinions

- ▶ Clever and concise.
- ▶ Technical proofs are quite poorly written.
- ▶ Is it possible to generalize this to weighted edges (without incurring polylog factors or practical costs)?
- ▶ In the parallel setting, are there known reasons to use weak diameter decompositions over strong diameter?



References

- ▶ Miller, Gary L., Richard Peng, and Shen Chen Xu. "Parallel graph decompositions using random shifts." Proceedings of the twenty-fifth annual ACM symposium on Parallelism in algorithms and architectures. 2013.
- ▶ Blelloch, Guy E., et al. "Near linear-work parallel SDD solvers, low-diameter decomposition, and low-stretch subgraphs." Proceedings of the twenty-third annual ACM symposium on Parallelism in algorithms and architectures. 2011.

An LLM was used to generated the Tikz figure and fix mistakes in the math.