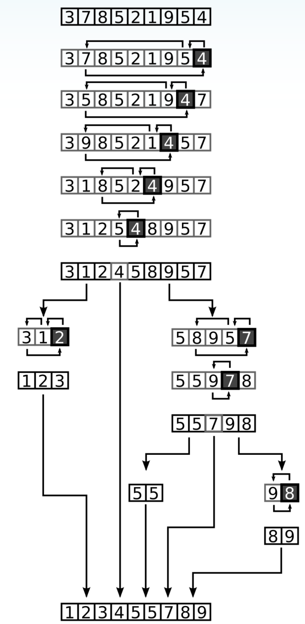
6.506: Algorithm Engineering



Lecture 2
Parallel Algorithms

Julian Shun

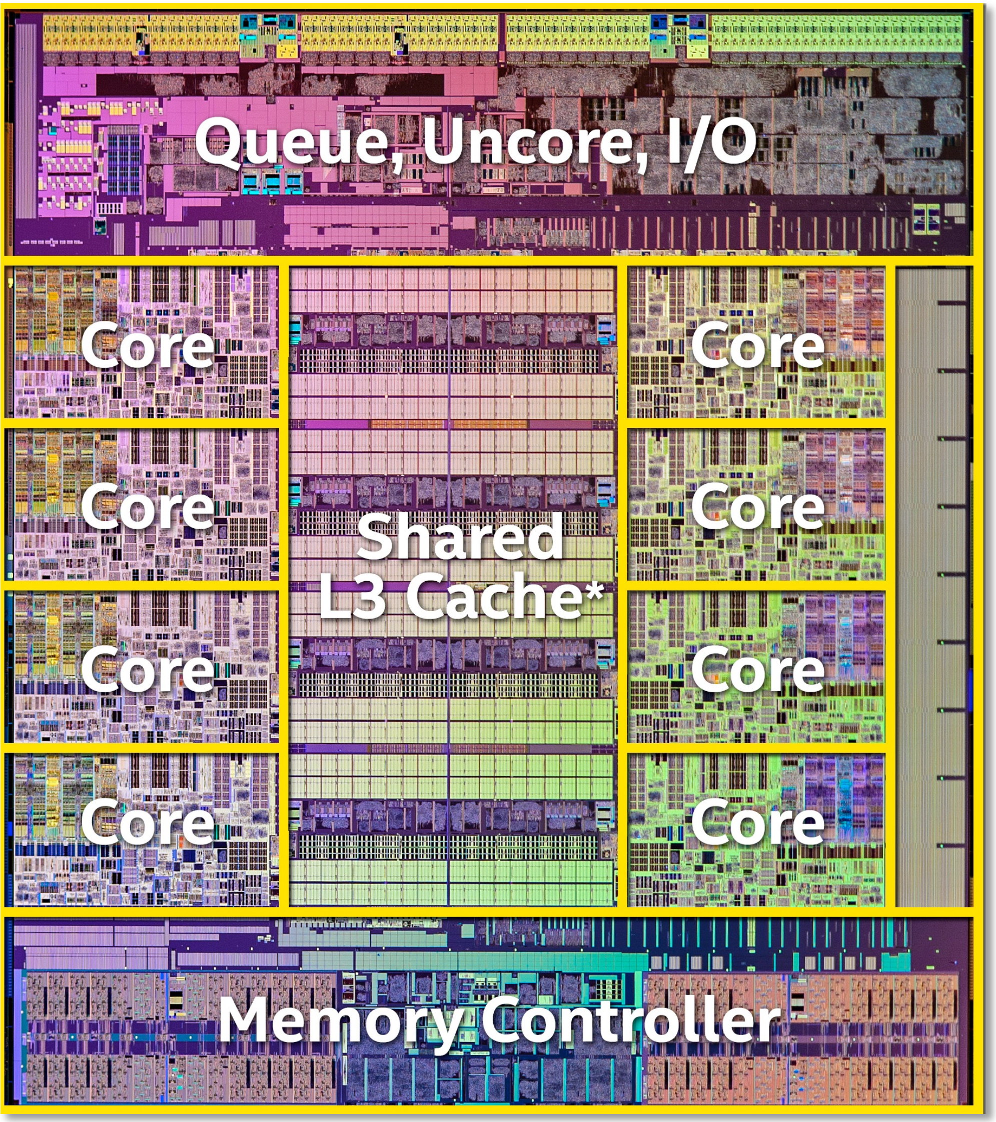
February 9, 2022

Lecture material taken from “Parallel Algorithms” by Guy Blelloch and Bruce Maggs and 6.172, developed by Charles Leiserson and Saman Amarasinghe

Announcement

- Problem set will be released on Canvas this week, due on Monday 3/6

Multicore Processors



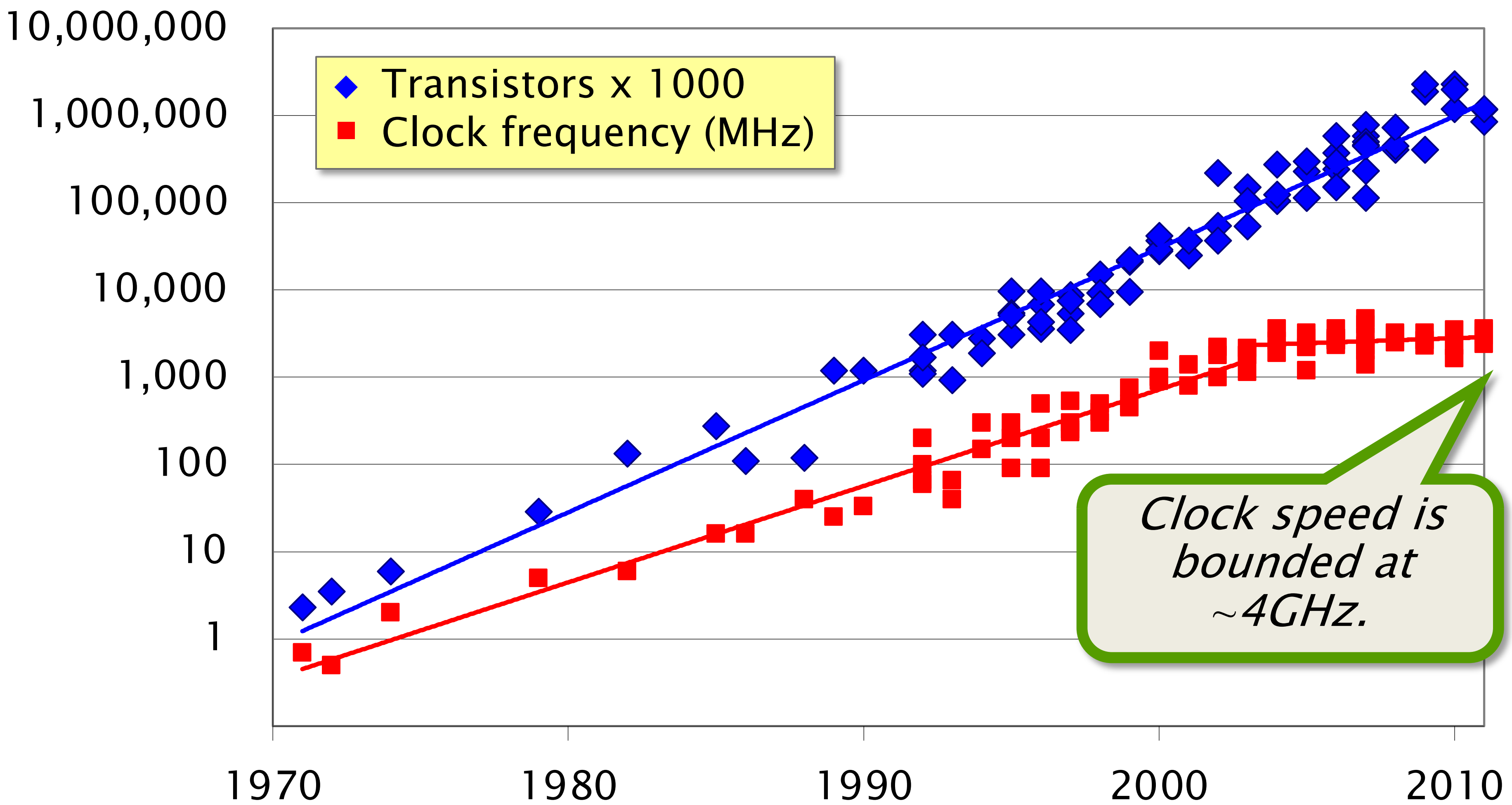
Q Why do semiconductor vendors provide chips with multiple processor cores?

A Because of Moore's Law and the end of the scaling of clock frequency.

Intel Haswell-E

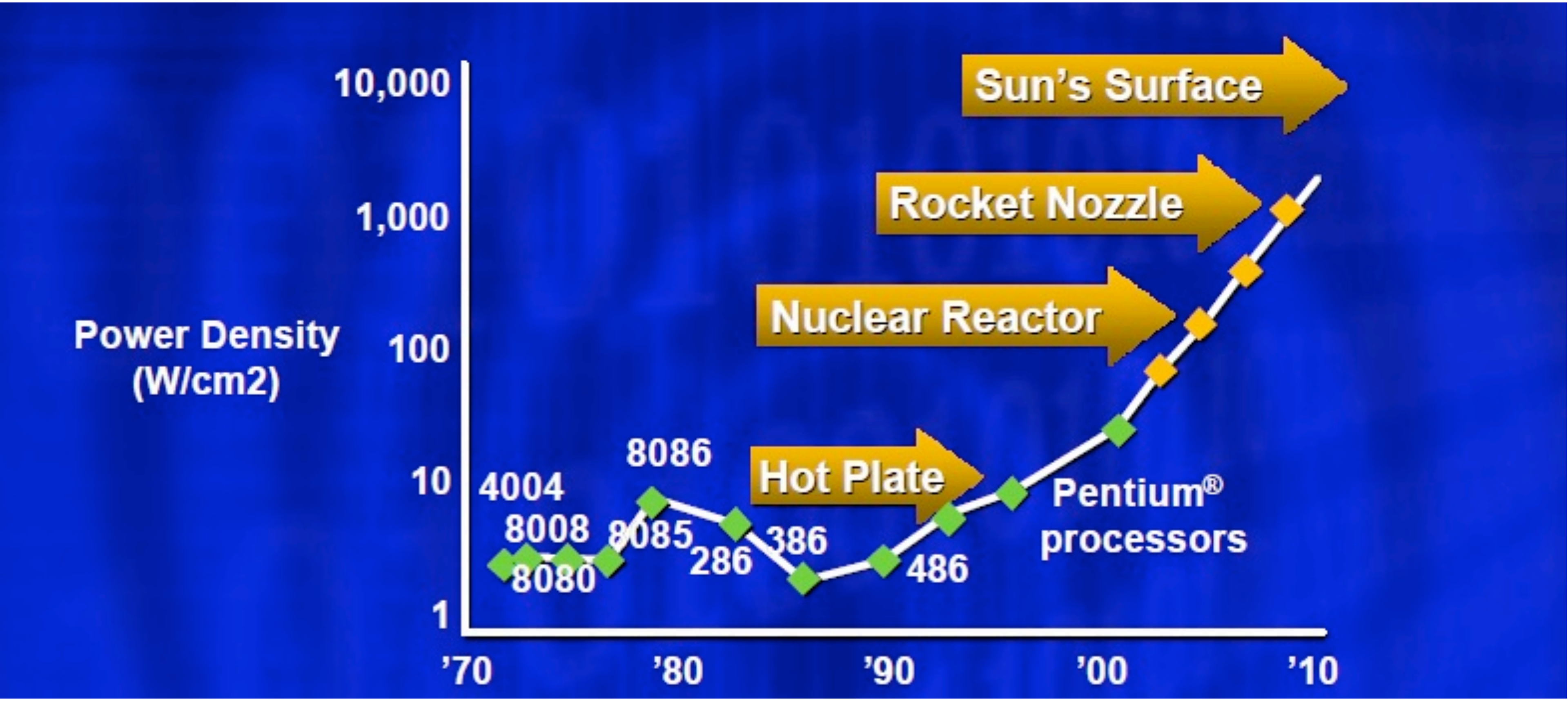
Slide adapted from 6.172 (Charles Leiserson and Saman Amarasinghe)

Technology Scaling



Slide adapted from 6.172 (Charles Leiserson and Saman Amarasinghe)

Power Density

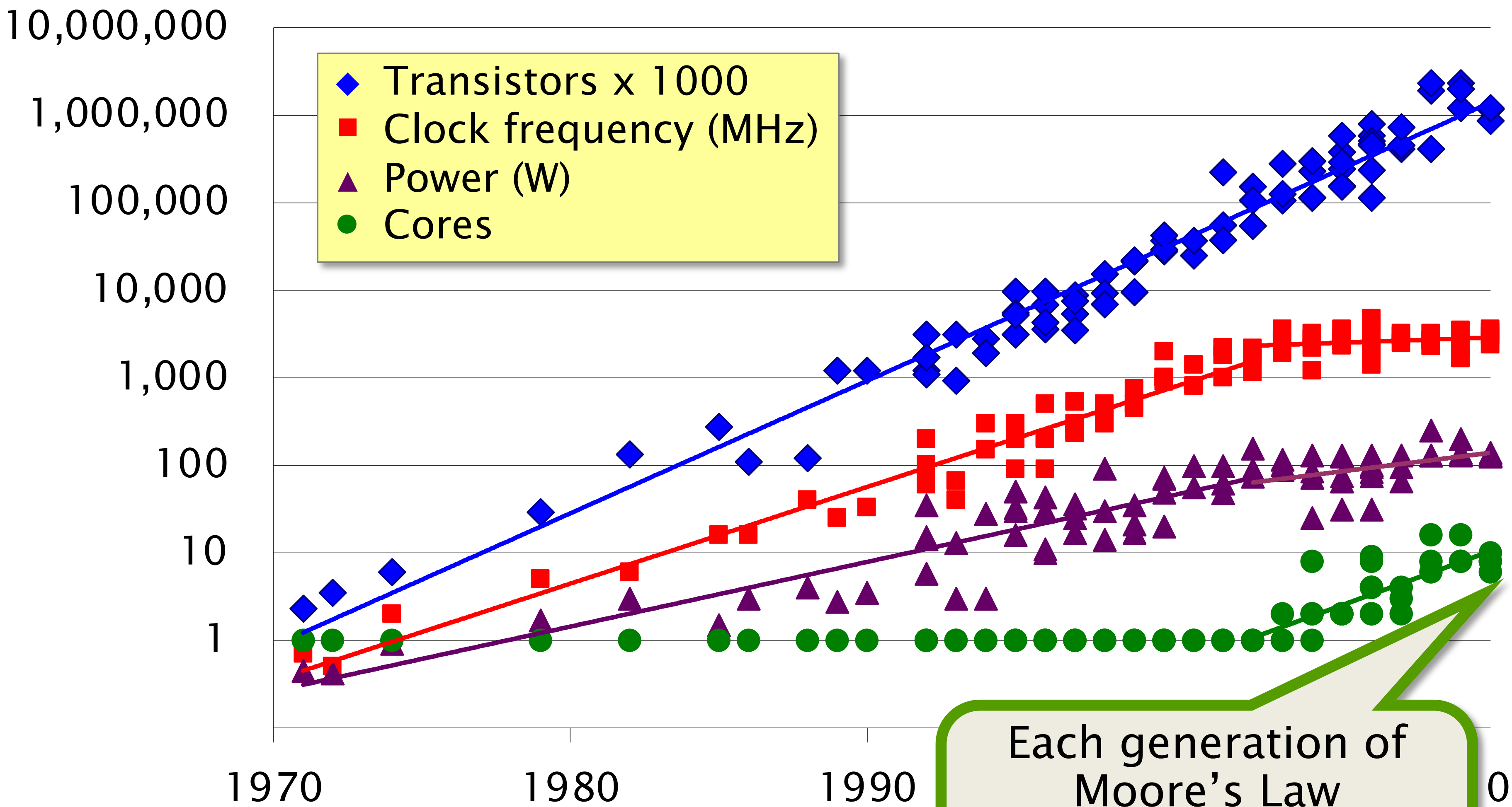


Source: Patrick Gelsinger, *Intel Developer's Forum*, Intel Corporation, 2004.

Projected power density, if clock frequency had continued its trend of scaling 25%–30% per year.

Slide adapted from 6.172 (Charles Leiserson and Saman Amarasinghe)

Technology Scaling



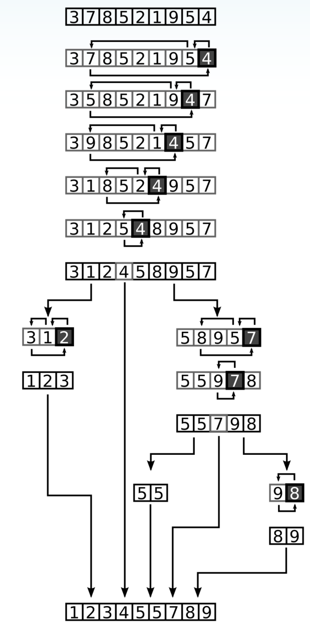
Each generation of Moore's Law potentially doubles the number of cores.

Slide adapted from 6.172 (Charles Leiserson and Saman Amarasinghe)

Parallel Languages

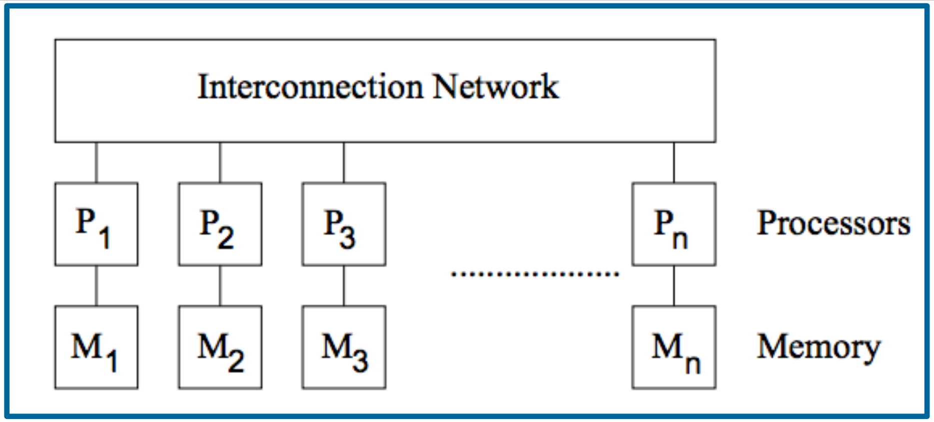
- Pthreads
- Cilk, OpenMP
- Message Passing Interface (MPI)
- CUDA, OpenCL

- Today: Shared-memory parallelism
  - Cilk and OpenMP are extensions of C/C++ that support parallel for-loops, parallel recursive calls, etc.
  - Do not need to worry about assigning tasks to processors as these languages have a runtime scheduler
  - Cilk has a provably efficient runtime scheduler

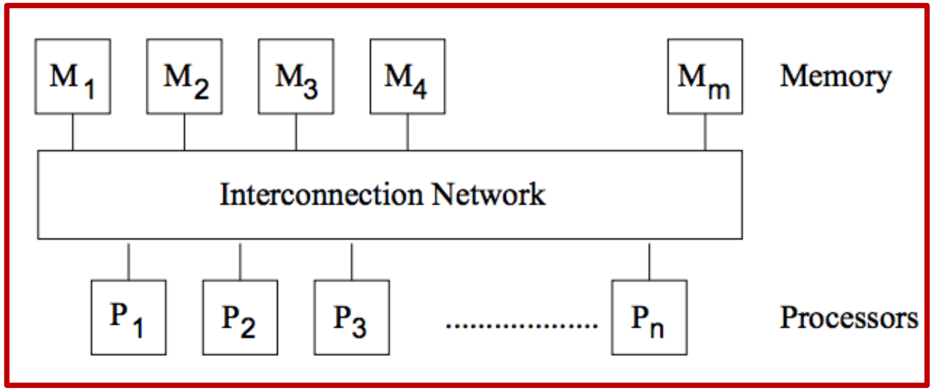


PARALLELISM MODELS

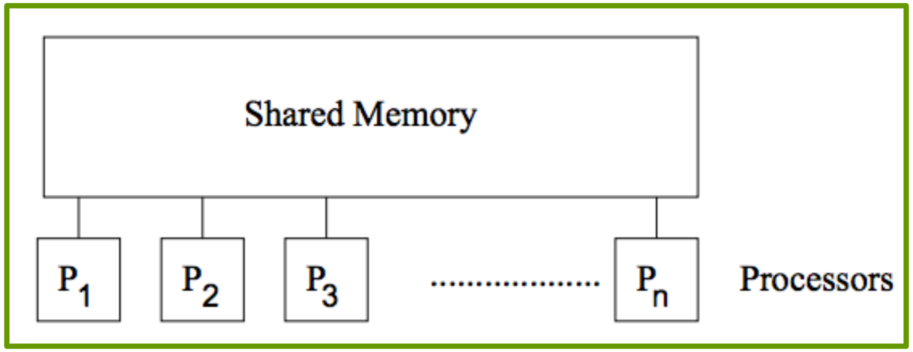
Basic multiprocessor models



Local memory machine



Modular memory machine

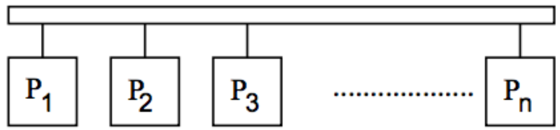


Parallel random-access Machine (PRAM)

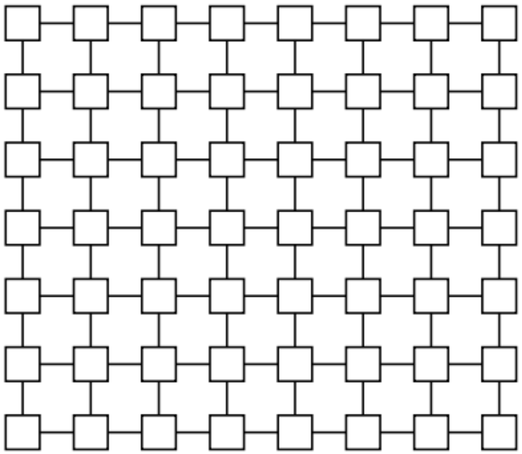
Source: "Parallel Algorithms" by Guy E. Blelloch and Bruce M. Maggs

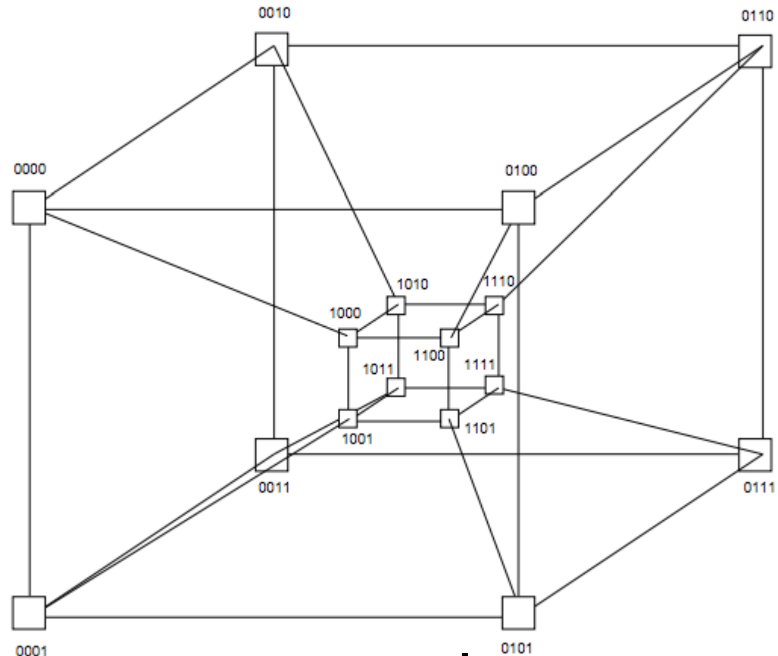
Network topology

Bus

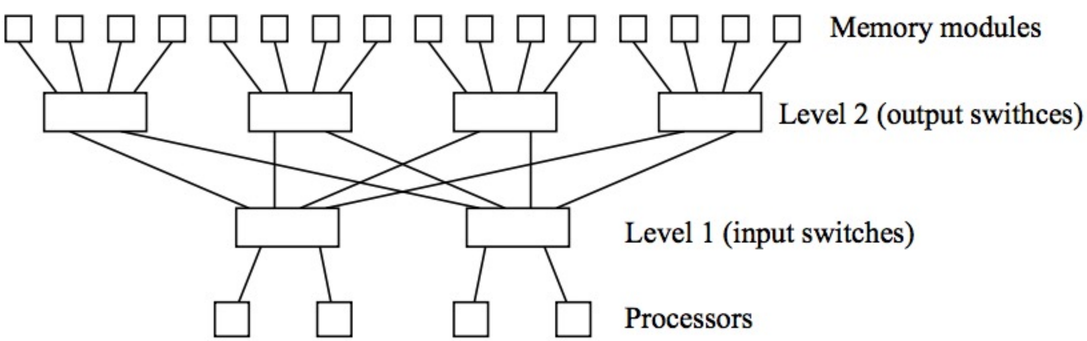


Mesh

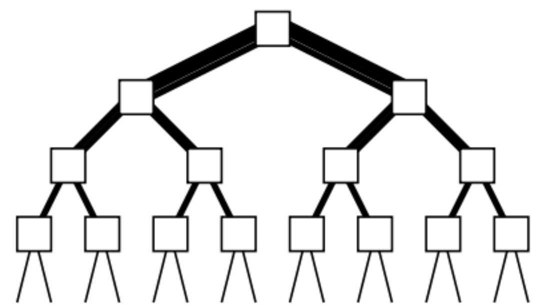




Hypercube



2-level multistage network



Fat tree

Source: “Parallel Algorithms” by Guy E. Blelloch and Bruce M. Maggs

Network topology

- Algorithms for specific topologies can be complicated
  - May not perform well on other networks
- Alternative: use a model that summarizes latency and bandwidth of network
  - Postal model
  - Bulk-Synchronous Parallel (BSP) model
  - LogP model

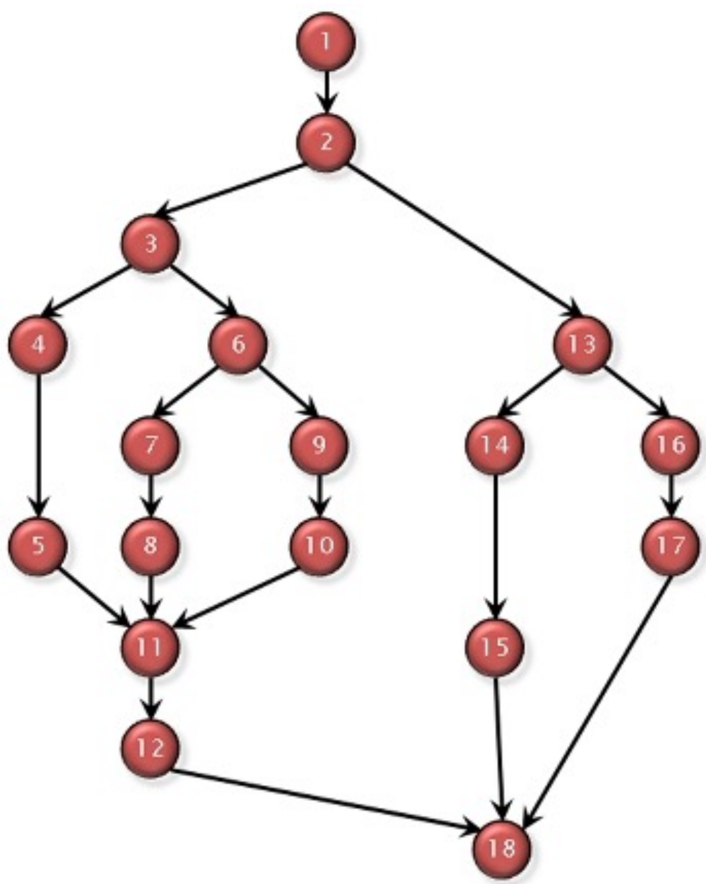
PRAM Model

- All processors can perform same local instructions as in the RAM model
- All processors operate in lock-step
- Implicit synchronization between steps
- Models for concurrent access
  - Exclusive-read exclusive-write (EREW)
  - Concurrent-read concurrent-write (CRCW)
    - How to resolve concurrent writes: arbitrary value, value from lowest-ID processor, logical OR of values, sum of values
  - Concurrent-read exclusive-write (CREW)
  - Queue-read queue-write (QRQW)
    - Allows concurrent access in time proportional to the maximal number of concurrent accesses

Work-Span model

- Similar to PRAM but does not require lock-step or processor allocation

Computation graph

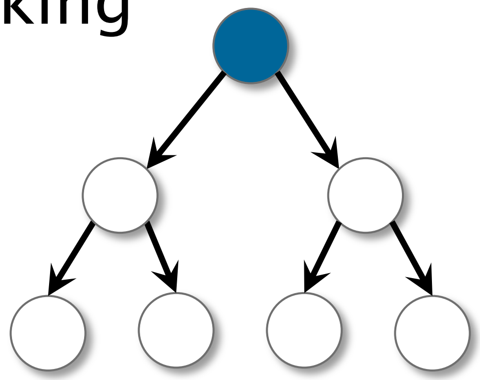


- Work = number of vertices in graph (number of operations)
- Span (Depth) = longest directed path in graph (dependence length)
- Parallelism = Work / Span
- A work-efficient parallel algorithm has work that asymptotically matches the best sequential algorithm for the problem

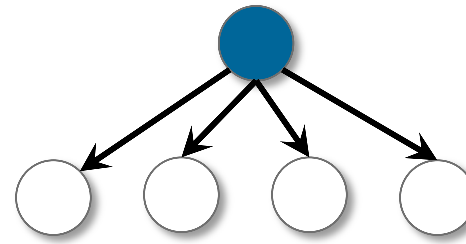
Goal: work-efficient and low (polylogarithmic) span parallel algorithms

Work–Span model

- Spawning/forking tasks
  - Model can support either binary forking or arbitrary forking



Binary forking



Arbitrary forking

  - Cilk uses binary forking, as seen in 6.172
  - Converting between the two changes work by at most a constant factor and span by at most a logarithmic factor
    - Keep this in mind when reading textbooks/papers on parallel algorithms
  - We will assume arbitrary forking unless specified

Work-Span model

- State what operations are supported
  - Concurrent reads/writes?
  - Resolving concurrent writes

Scheduling

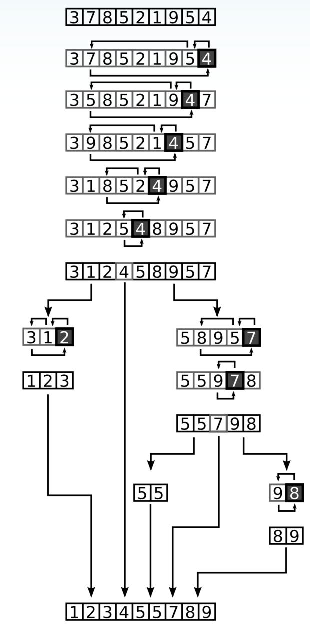
- For a computation with work W and span S, on P processors a greedy scheduler achieves

Running time $\leq W/P + S$

- For a computation with work W and span S, on P processors Cilk's work-stealing scheduler achieves

Expected running time $\leq W/P + O(S)$

- Work-efficiency is important since P and S are usually small



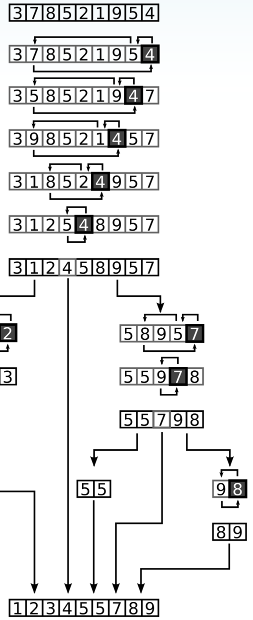
PARALLEL SUM

Parallel Sum

- Definition: Given a sequence $A=[x_0, x_1,\ldots, x_{n-1}]$, return $x_0+x_1+\ldots+x_{n-2}+x_{n-1}$

What is the span?
$S(n) = S(n/2)+O(1)$
$S(1) = O(1)$
$\rightarrow S(n) = O(\log n)$

What is the work?
$W(n) = W(n/2)+O(n)$
$W(1) = O(1)$
$\rightarrow W(n) = O(n)$


3 7 8 5 2 1 9 5 4
3 7 8 5 2 1 9 5 4
3 5 8 5 2 1 9 4 7
3 9 8 5 2 1 4 5 7
3 1 8 5 2 4 9 5 7
3 1 2 5 4 8 9 5 7
3 1 2 4 5 8 9 5 7
3 1 2
5 8 9 5 7
1 2 3
5 5 9 7 8
5 5 7 9 8
5 5
9 8
8 9
1 2 3 4 5 5 7 8 9

PREFIX SUM

Prefix Sum

- Definition: Given a sequence $A=[x_0, x_1, \ldots, x_{n-1}]$, return a sequence where each location stores the sum of everything before it in A, $[0, x_0, x_0+x_1, \ldots, x_0+x_1+\ldots+x_{n-2}]$, as well as the total sum $x_0+x_1+\ldots+x_{n-2}+x_{n-1}$

- Example:

2	4	3	1	3

0	2	6	9	10

Total sum = 13

- Can be generalized to any associative binary operator (e.g., $\times$, min, max)

Sequential Prefix Sum

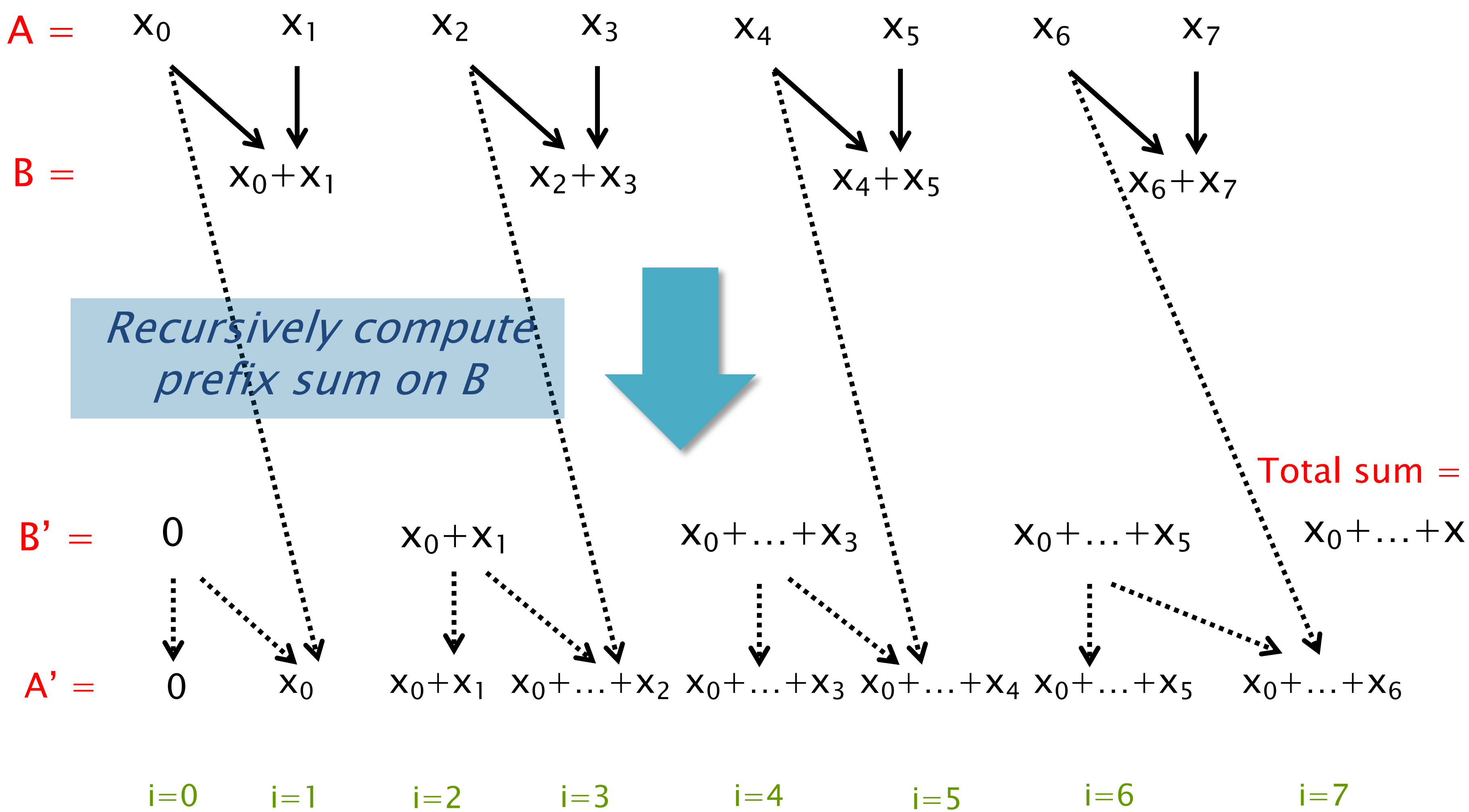
Input: array A of length n

Output: array A’ and total sum

```
cumulativeSum = 0;
for i=0 to n-1:
   A’[i] = cumulativeSum;
   cumulativeSum += A[i];
return A’ and cumulativeSum
```

- What is the work of this algorithm?
  - O(n)
- Can we execute iterations in parallel?
  - Loop carried dependence: value of cumulativeSum depends on previous iterations

Parallel Prefix Sum



Total sum = $x_0+\ldots+x_7$

for even values of i: A'[i] = B'[i/2]

for odd values of i: A'[i] = B'[(i–1)/2]+A[i–1]

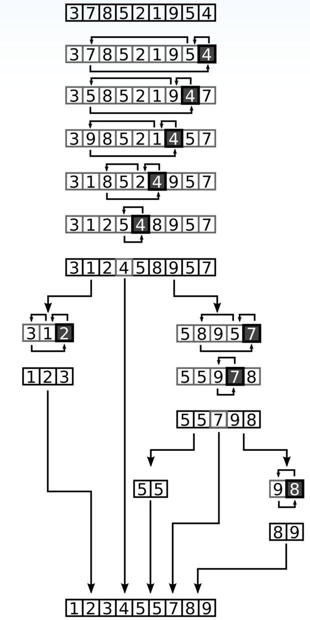
Parallel Prefix Sum

Input: array A of length n (assume n is a power of 2)
Output: array A' and total sum

```
PrefixSum(A, n):
  if n == 1: return ([0], A[0])
  for i=0 to n/2-1 in parallel:
    B[i] = A[2i] + A[2i+1]
  (B', sum) = PrefixSum(B, n/2)
  for i=0 to n-1 in parallel:
    if (i mod 2) == 0:   A'[i] = B'[i/2]
    else:   A'[i] = B'[(i-1)/2] + A[i-1]
  return (A', sum)
```

What is the span?
$S(n) = S(n/2)+O(1)$
$S(1) = O(1)$
→ $S(n) = O(\log n)$

What is the work?
$W(n) = W(n/2)+O(n)$
$W(1) = O(1)$
→ $W(n) = O(n)$


3 7 8 5 2 1 9 5 4
3 7 8 5 2 1 9 5 4
3 5 8 5 2 1 9 4 7
3 9 8 5 2 1 4 5 7
3 1 8 5 2 4 9 5 7
3 1 2 5 4 8 9 5 7
3 1 2 4 5 8 9 5 7
3 1 2
5 8 9 5 7
1 2 3
5 5 9 7 8
5 5 7 9 8
5 5
9 8
8 9
1 2 3 4 5 5 7 8 9

FILTER

Filter

- Definition: Given a sequence $A=[x_0, x_1,\ldots, x_{n-1}]$ and a Boolean array of flags $B[b_0, b_1,\ldots, b_{n-1}]$, output an array A’ containing just the elements A[i] where B[i] = true (maintaining relative order)
- Example:

A =

2	4	3	1	3

B =

T	F	T	T	F

A’ =

2	3	1

- Can you implement filter using prefix sum?

Filter Implementation

A =

2	4	3	1	3

B =

T	F	T	T	F

1	0	1	1	0

Prefix sum

B' =

0	1	1	2	3

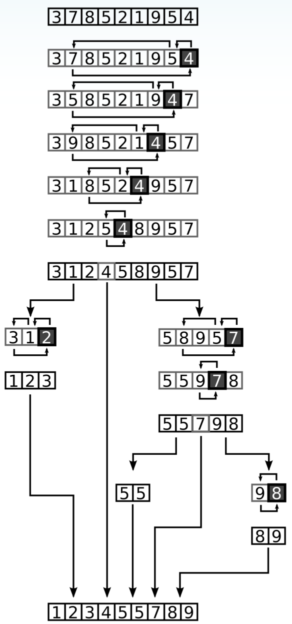
Total sum = 3

```
//Assume B'[n] = total sum
parallel-for i=0 to n-1:
    if(B'[i] != B'[i+1]):
        A'[B'[i]] = A[i];
```

Allocate array of size 3

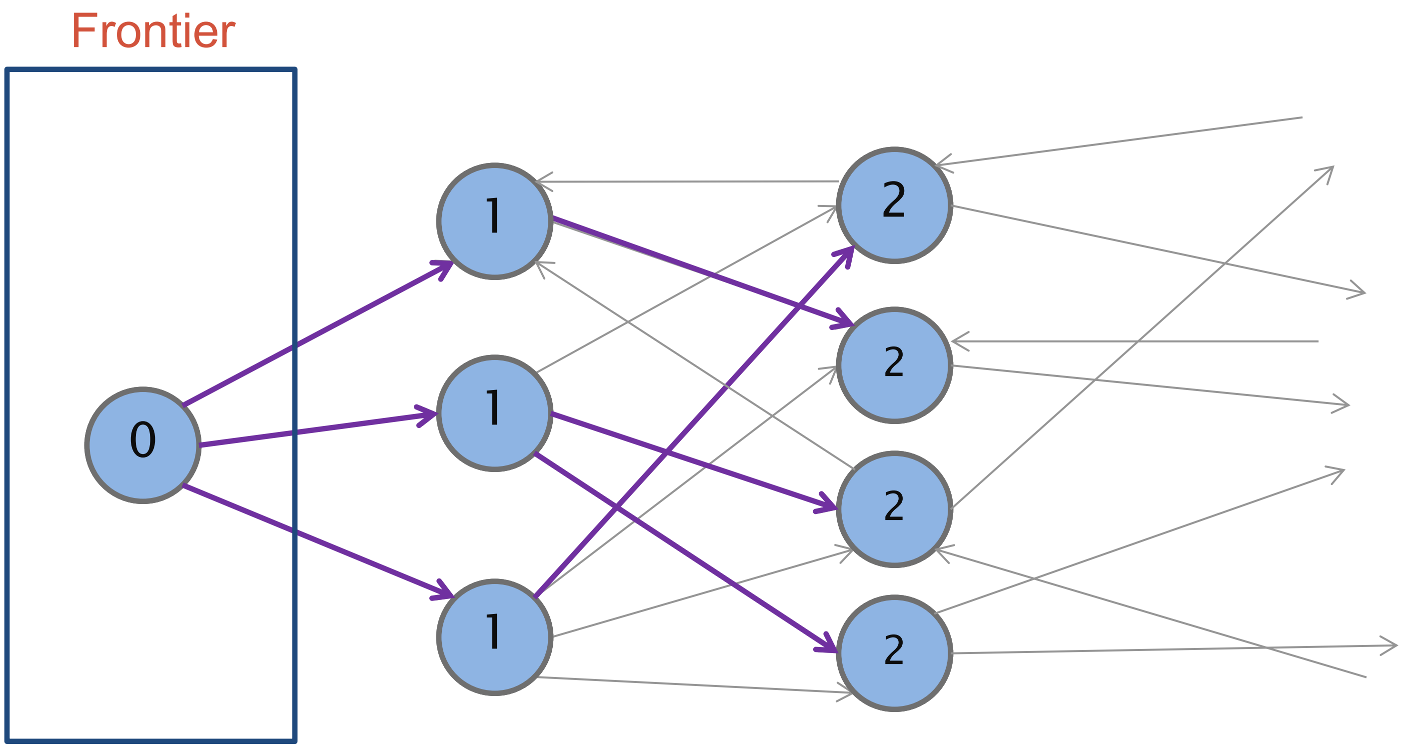
A' =

2	3	1



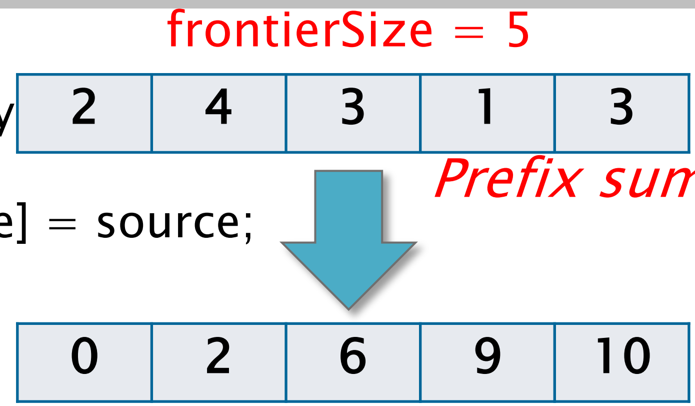
PARALLEL BREADTH-FIRST SEARCH

Parallel BFS Algorithm



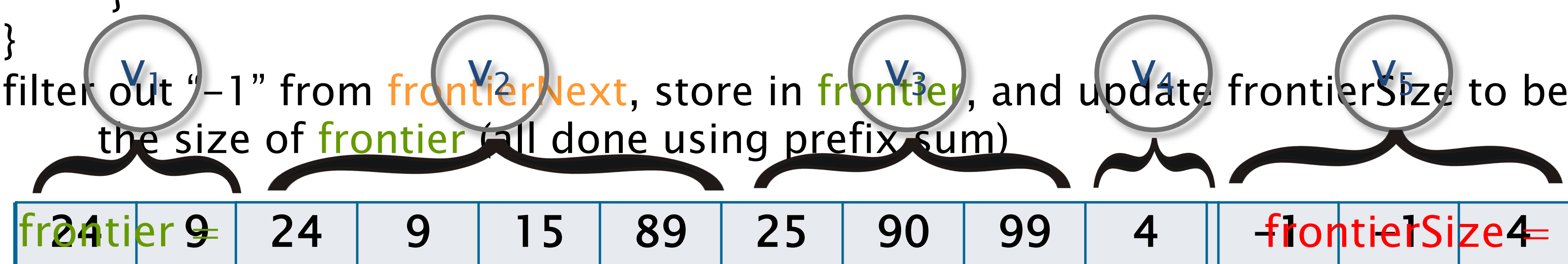
- Can process each frontier in parallel
  - Parallelize over both the vertices and their outgoing edges

Parallel BFS Code




```
BFS(Offsets, Edges, source) {
  parent, frontier, frontierNext, and degrees are arrays
  parallel_for(int i=0; i<n; i++) parent[i] = -1;
  frontier[0] = source, frontierSize = 1, parent[source] = source;

  while(frontierSize > 0) {
    parallel_for(int i=0; i<frontierSize; i++)
        degrees[i] = Offsets[frontier[i]+1] – Offsets[frontier[i]];
    perform prefix sum on degrees array
    parallel_for(int i=0; i<frontierSize; i++) {
        v = frontier[i], index = degrees[i], d = Offsets[v+1]-Offsets[v];
        for(int j=0; j<d; j++) { //can be parallel
            ngh = Edges[Offsets[v]+j];
            if(parent[ngh] == -1 && compare-and-swap(&parent[ngh], -1, v)) {
              frontierNext[index+j] = ngh;
            } else { frontierNext[index+j] = -1; }
        }
    }
    filter out "-1" from frontierNext, store in frontier, and update frontierSize to be
        the size of frontier (all done using prefix sum)
  }
}
```



BFS Work-Span Analysis

- Number of iterations <= diameter Δ of graph
- Each iteration takes O(log m) span for prefix sum and filter (assuming inner loop is parallelized)

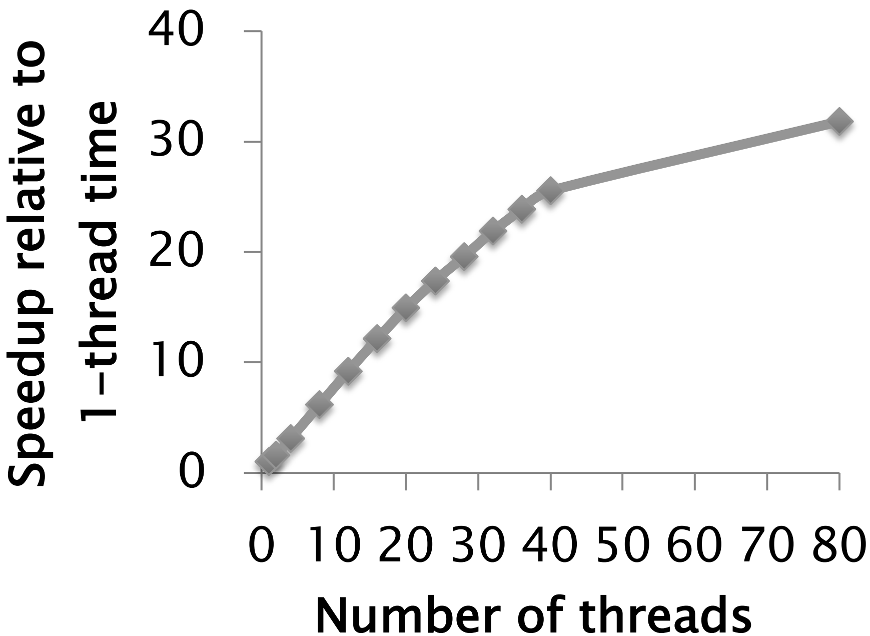
Span = O(Δ log m)

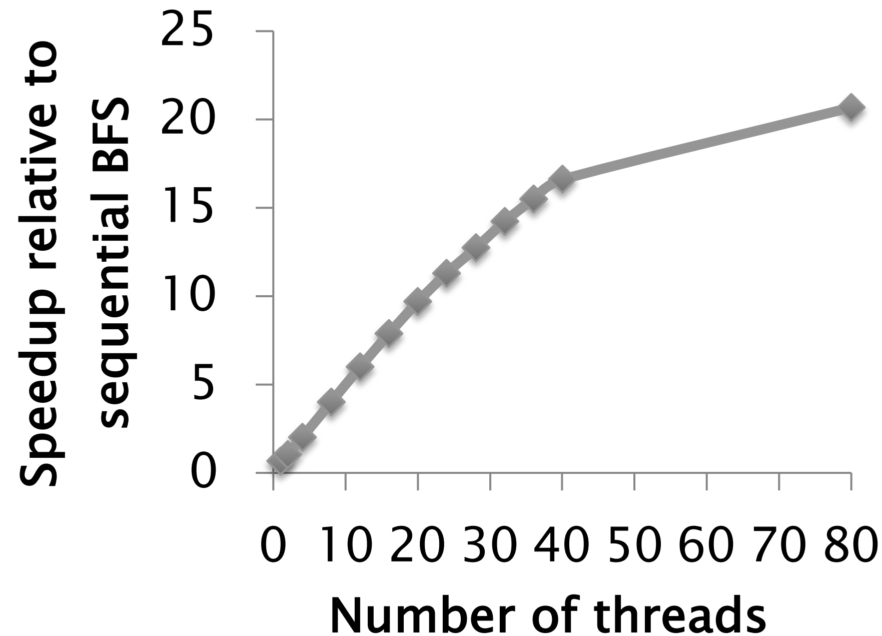
- Sum of frontier sizes = n
- Each edge traversed once -> m total visits
- Work of prefix sum on each iteration is proportional to frontier size -> Θ(n) total
- Work of filter on each iteration is proportional to number of edges traversed -> Θ(m) total

Work = Θ(n+m)

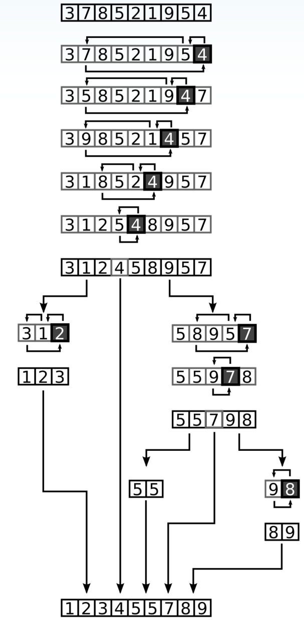
Performance of Parallel BFS

- Random graph with $n=10^7$ and $m=10^8$
  - 10 edges per vertex
- 40-core machine with 2-way hyperthreading





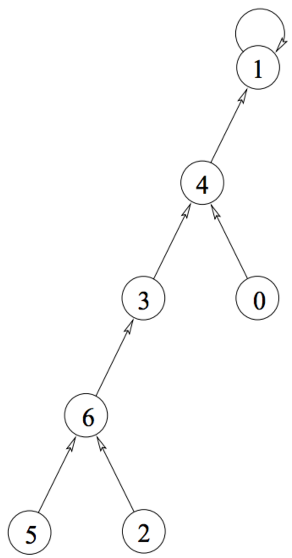
- 31.8x speedup on 40 cores with hyperthreading
- Sequential BFS is 54% faster than parallel BFS on 1 thread



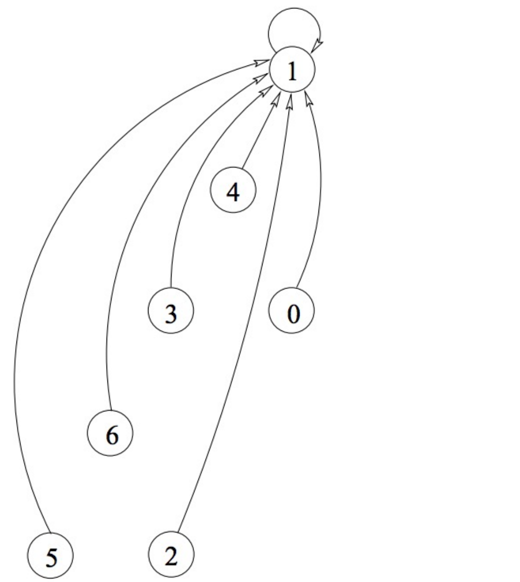
POINTER JUMPING AND LIST RANKING

Pointer Jumping

- Have every node in linked list or rooted tree point to the end (root)



(a) The input tree $P = [4, 1, 6, 4, 1, 6, 3]$.



(b) (c) The final tree $P = [1, 1, 1, 1, 1, 1, 1]$. iteration

```
for j=0 to ceil(log n)-1:
    parallel-for i=0 to n-1:
        temp[i] = P[P[i]];
    parallel-for i=0 to n-1:
        P[i] = temp[i];
```

What is the work and span?

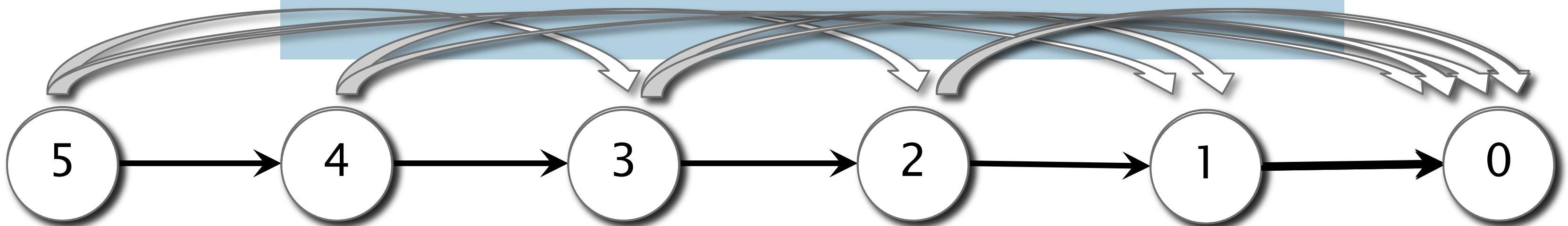
W = O(n log n)
S = O(log n)

List Ranking

- Have every node in linked list determine its distance to the end

```
parallel-for i=0 to n-1:
    if P[i] == i then rank[i] = 0
    else rank[i] = 1

for j=0 to ceil(log n)-1:
    temp, temp2;
    parallel-for i=0 to n-1:
        temp[i] = rank[P[i]];
        temp2[i] = P[P[i]];
    parallel-for i=0 to n-1:
        rank[i] = rank[i] + temp[i];
        P[i] = temp2[i];
```



Work–Span Analysis

```
parallel-for i=0 to n-1:
    if P[i] == i then rank[i] = 0
    else rank[i] = 1

for j=0 to ceil(log n)-1:
    temp, temp2;
    parallel-for i=0 to n-1:
        temp = rank[P[i]];
        temp2 = P[P[i]];
    parallel-for i=0 to n-1:
        rank[i] = rank[i] + temp;
        P[i] = temp2;
```

What is the work and span?

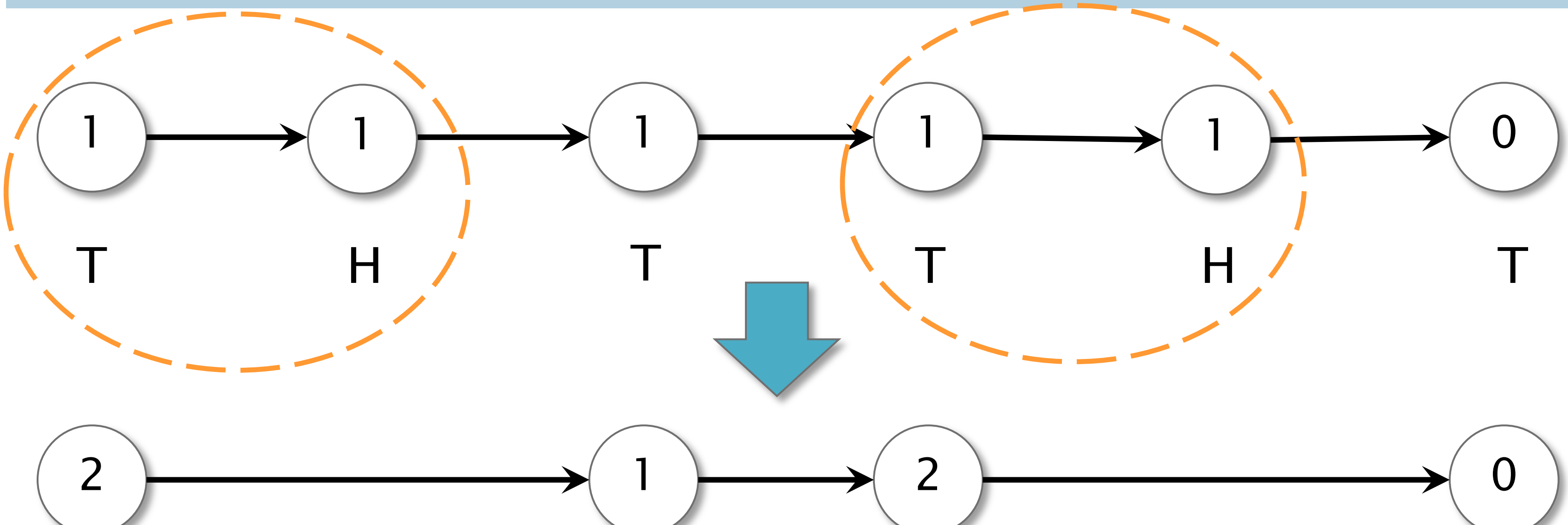
$W = O(n \log n)$
$S = O(\log n)$

Sequential algorithm only requires O(n) work

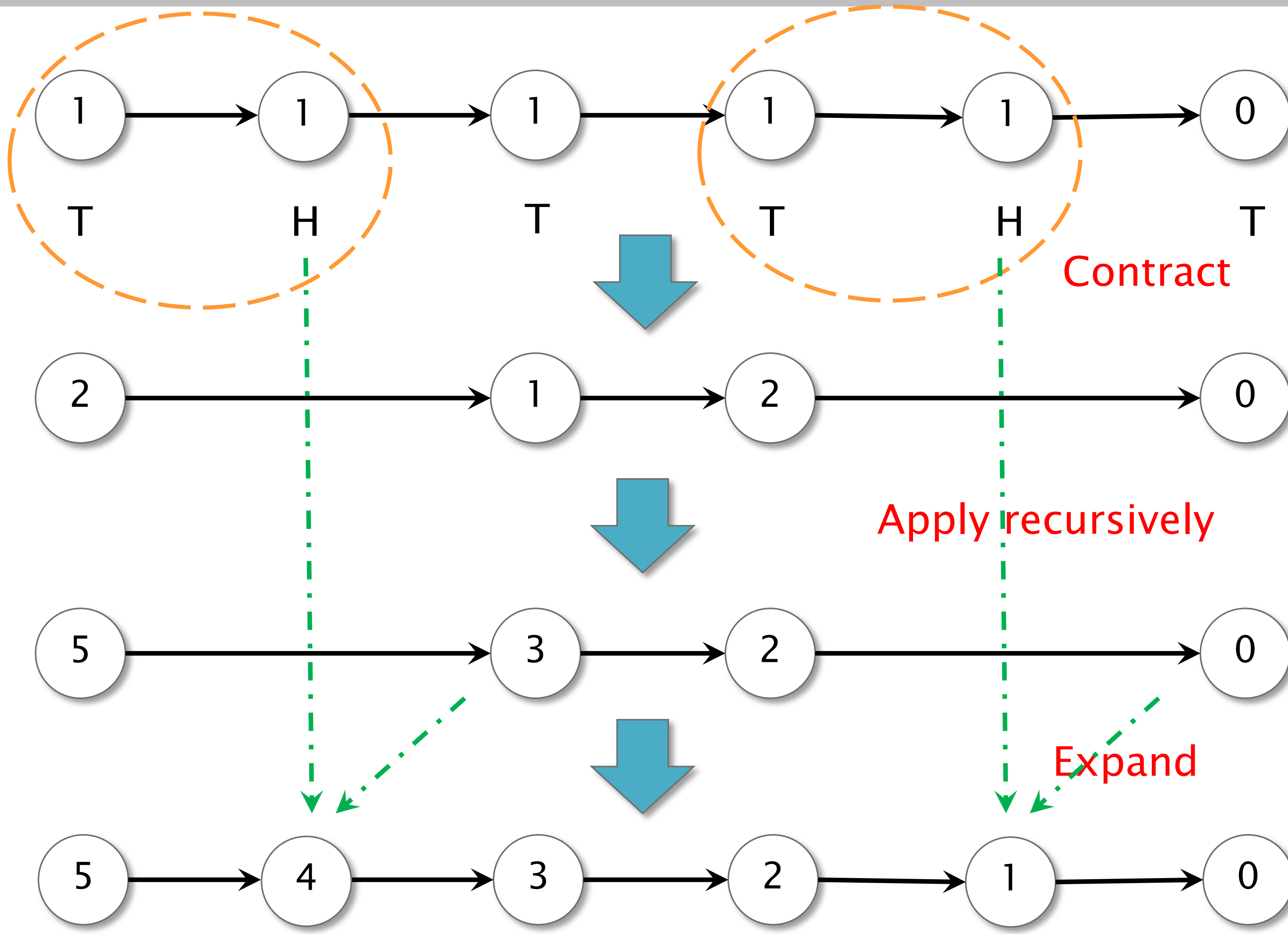
Work-Efficient List Ranking

ListRanking(list P)

1. If list has two or fewer nodes, then return //base case
2. Every node flips a fair coin
3. For each vertex u (except the last vertex), if u flipped Tails and P[u] flipped Heads then u will be paired with P[u]
   A. rank[u] = rank[u]+rank[P[u]]
   B. P[u] = P[P[u]]
4. Recursively call ListRanking on smaller list
5. Insert contracted nodes v back into list with rank[v] = rank[v] + rank[P[v]]



Work-Efficient List Ranking



Work-Span Analysis

- Number of pairs per round is $(n-1)/4$ in expectation
  - For all nodes u except for the last node, probability of u flipping Tails and P[u] flipping Heads is $1/4$
  - Linearity of expectations gives $(n-1)/4$ pairs overall
- Each round takes linear work and $O(1)$ span
- Expected work: $W(n) \leq W(7n/8) + O(n)$
- Expected span: $S(n) \leq S(7n/8) + O(1)$

$$W = O(n)$$
$$S = O(\log n)$$

- Can show span with high probability with Chernoff bound