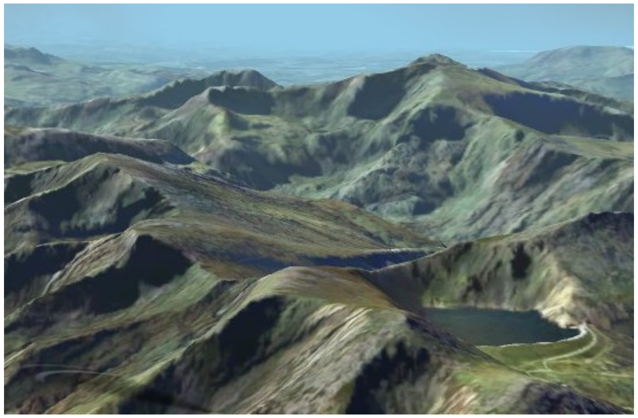
6.886 Delaunay Triangulation

Slides Credit: UFL COT5520, CIS4930 Spring 18
with minor modifications
Reference: Computational Geometry, Algorithms and Applications, 3rd Edition
yiqiuw@mit.edu

Motivation: Terrains

- a terrain is the graph of a function $f : A \subset \mathbb{R}^2 \to \mathbb{R}$
- we know only height values for a set of measurement points
- how can we interpolate the height at other points?
- using a triangulation



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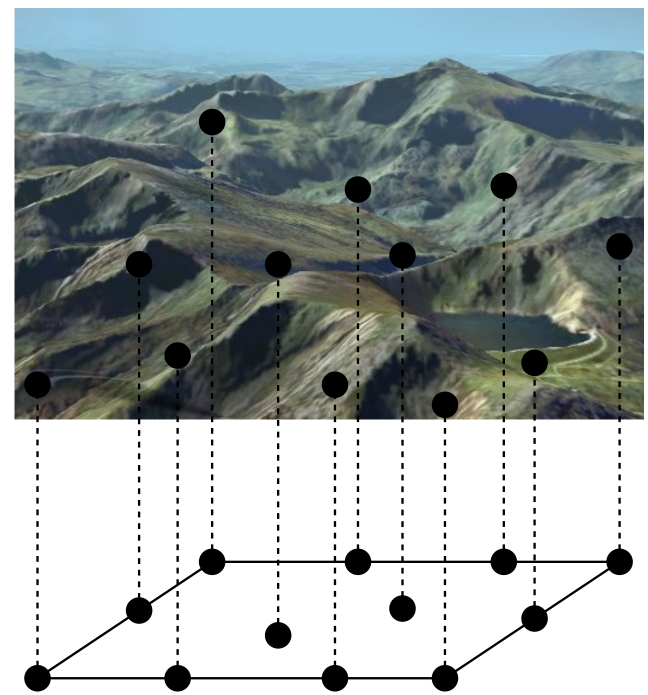

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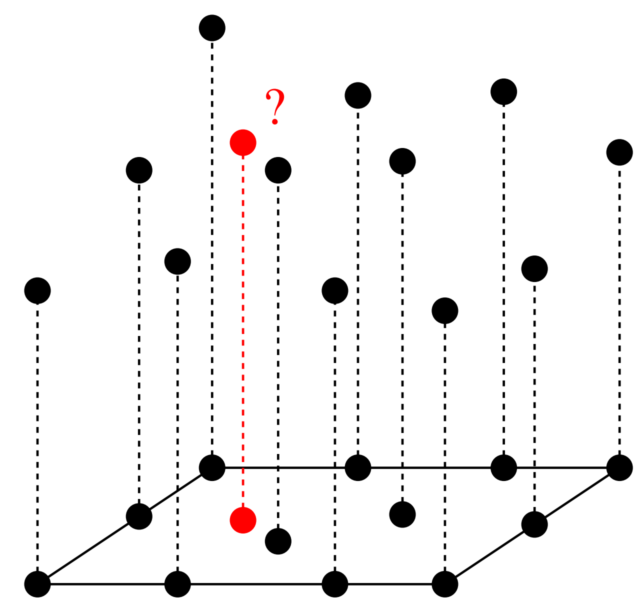

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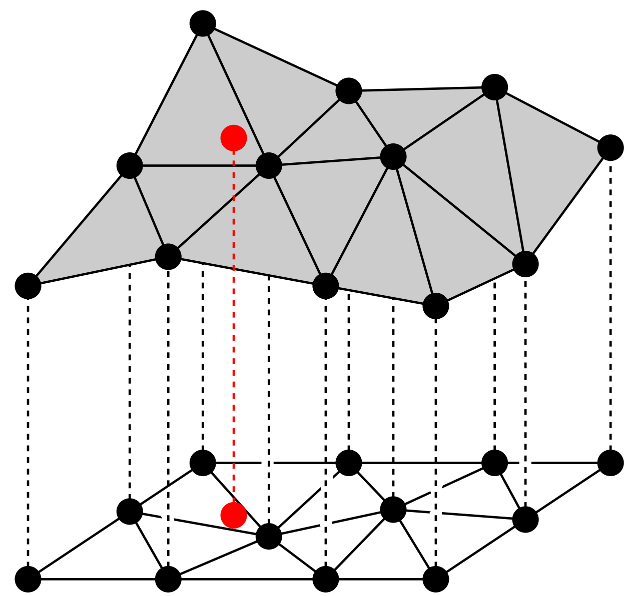

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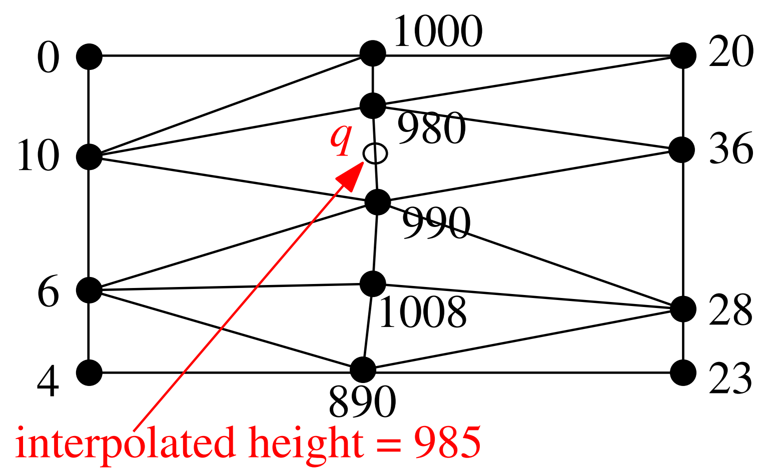
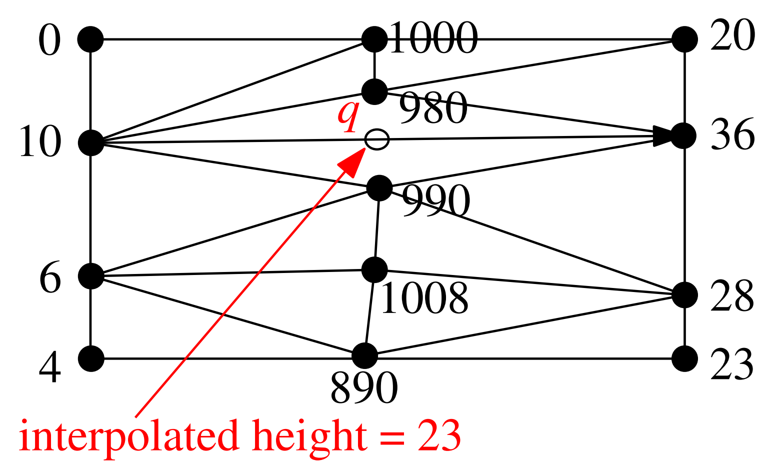
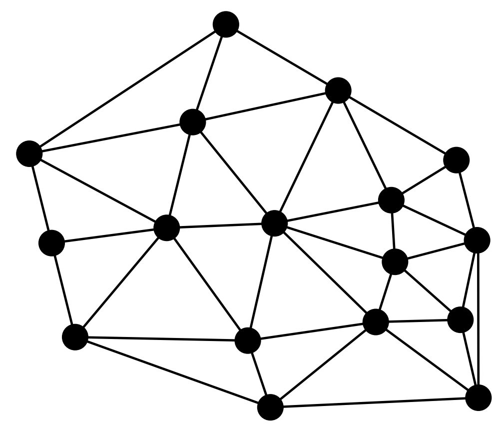
- a terrain is the graph of a function $f : A \subset \mathbb{R}^2 \rightarrow \mathbb{R}$
- we know only height values for a set of measurement points
- how can we interpolate the height at other points?
- using a triangulation
  – but which?





Triangulation

Let $P = \{p_1, \ldots, p_n\}$ be a point set. A triangulation of P is a maximal planar subdivision with vertex set P.



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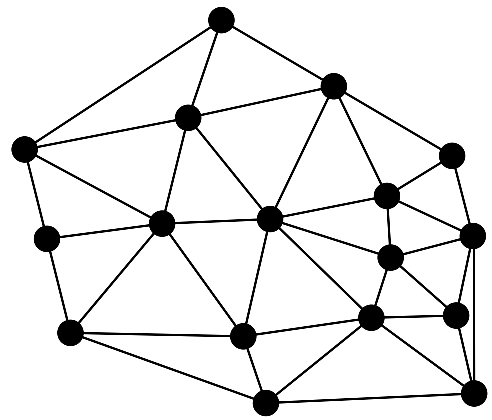
Complexity:

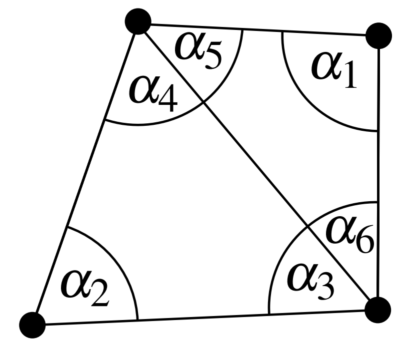
- $2n - 2 - k$ triangles
- $3n - 3 - k$ edges

where k is the number of points in P on the convex hull of P.



Angle Vector of a Triangulation

- Let $\mathcal{T}$ be a triangulation of P with m triangles and $3m$ vertices. Its angle vector is $A(\mathcal{T}) = (\alpha_1, \ldots, \alpha_{3m})$ where $\alpha_1, \ldots, \alpha_{3m}$ are the angles of $\mathcal{T}$ sorted by increasing value.
- Let $\mathcal{T}'$ be another triangulation of P. We define $A(\mathcal{T}) > A(\mathcal{T}')$ if $A(\mathcal{T})$ is lexicographically larger than $A(\mathcal{T}')$.
- $\mathcal{T}$ is angle optimal if $A(\mathcal{T}) \geq A(\mathcal{T}')$ for all triangulations $\mathcal{T}'$ of P.



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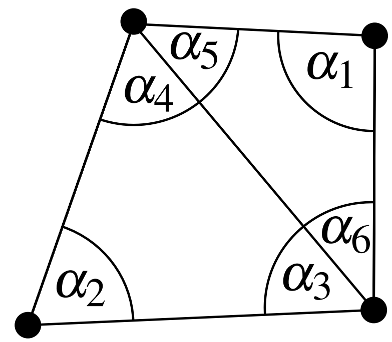

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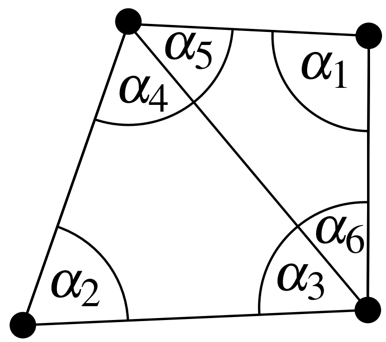

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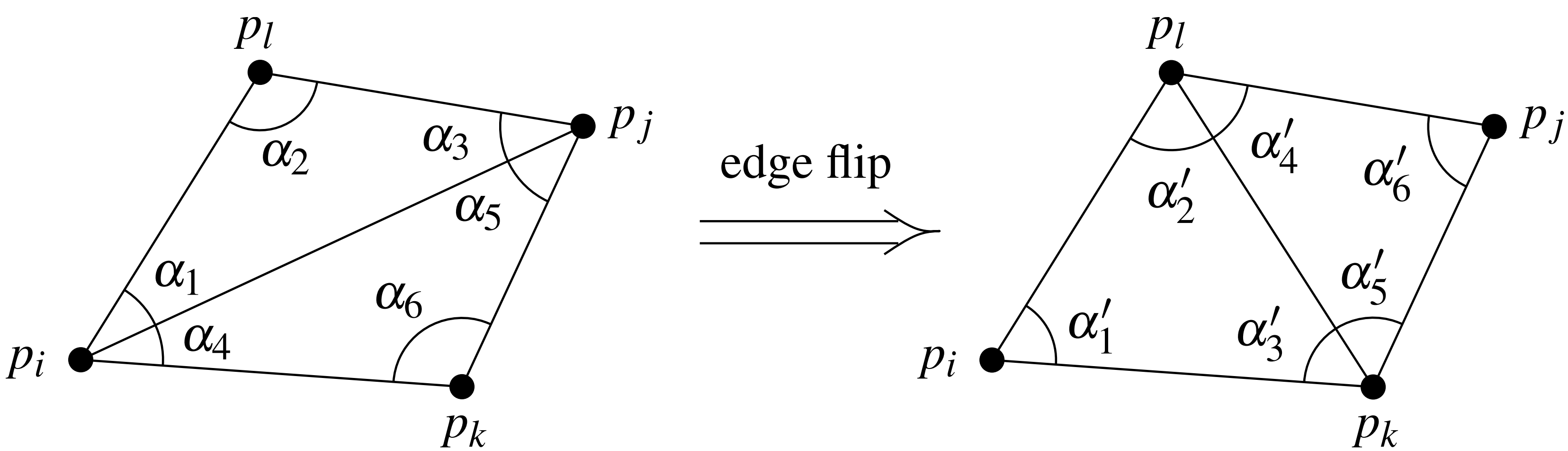
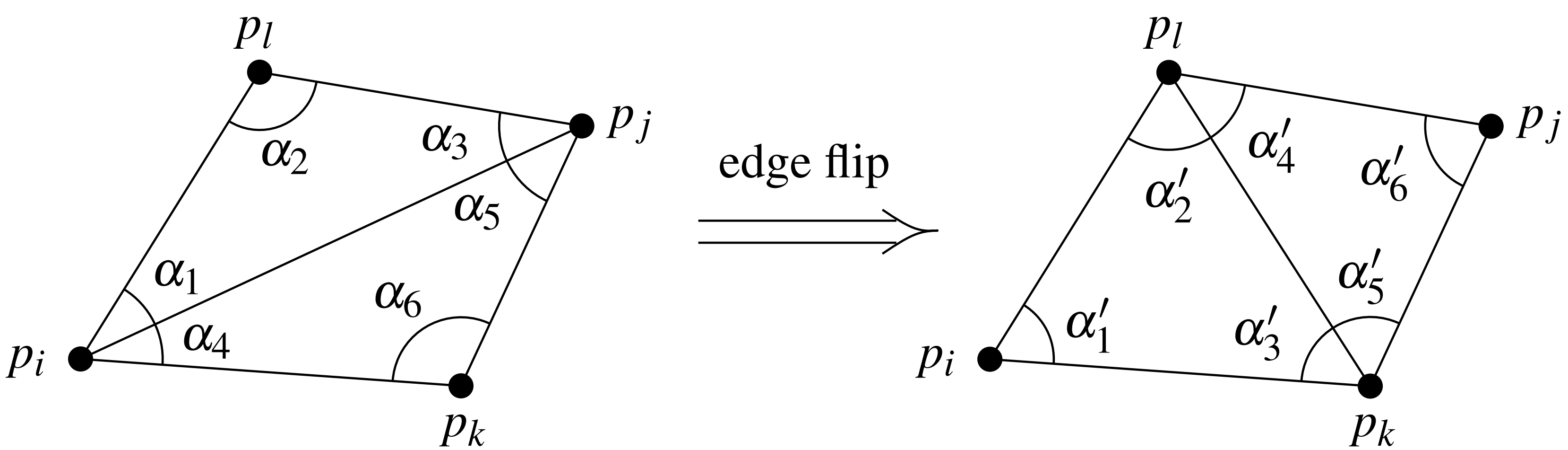
Edge Flipping



- Change in angle vector: $\alpha_1, \ldots, \alpha_6$ are replaced by $\alpha'_1, \ldots, \alpha'_6$.
- The edge $e = \overline{p_i p_j}$ is illegal if $\min_{1 \leq i \leq 6} \alpha_i < \min_{1 \leq i \leq 6} \alpha'_i$.
- Flipping an illegal edge increases the angle vector.

Edge Flipping



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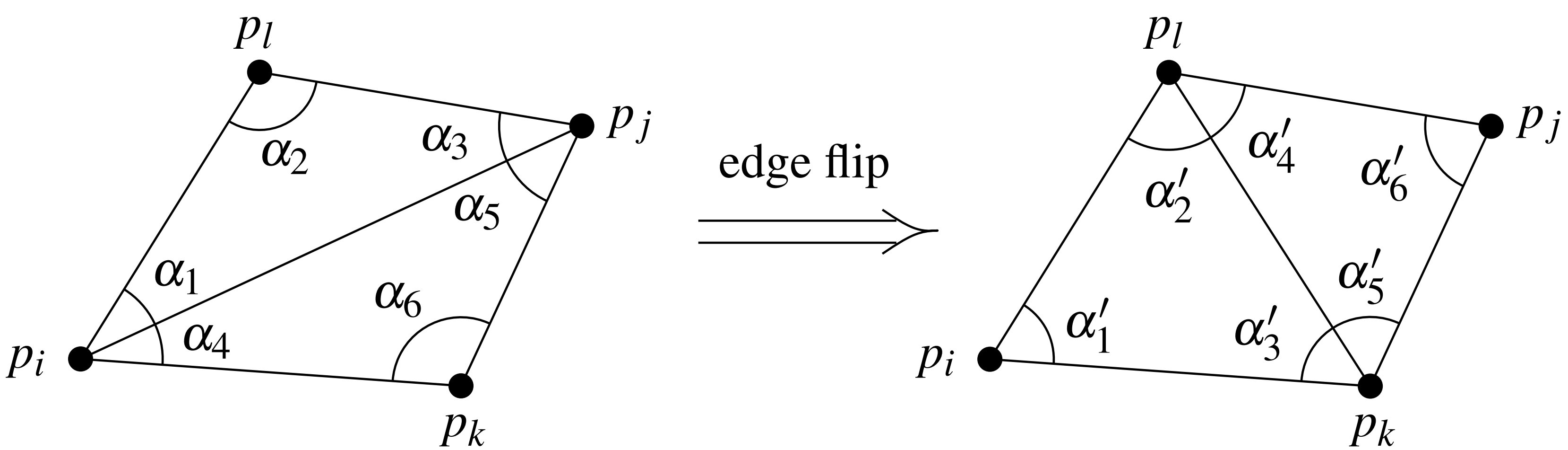

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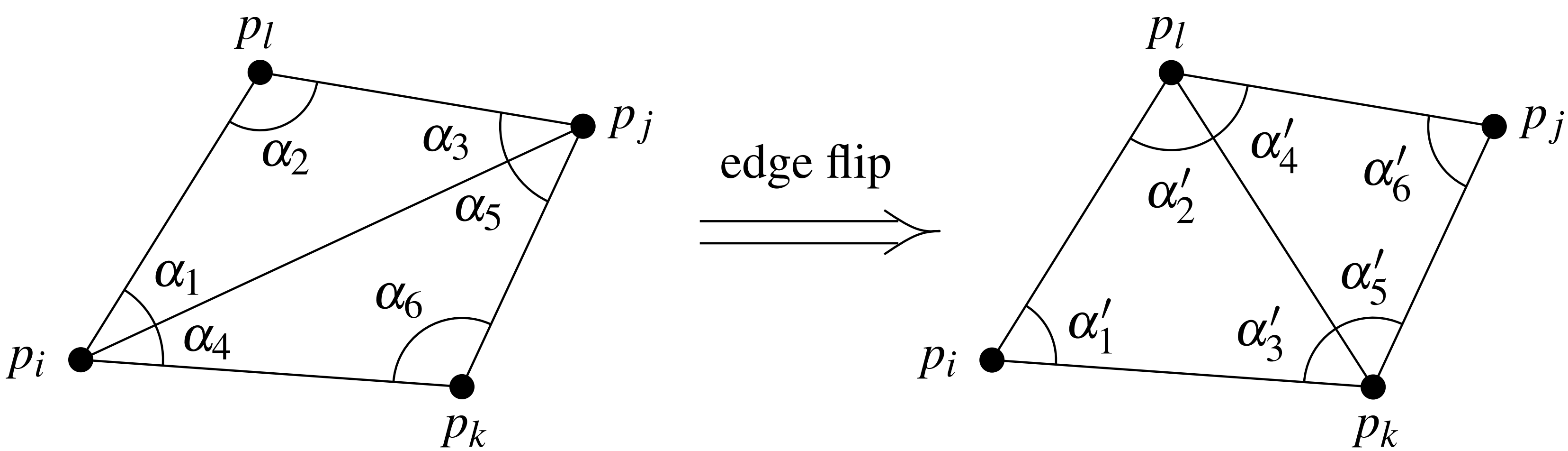

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Characterisation of Illegal Edges

How do we determine if an edge is illegal?

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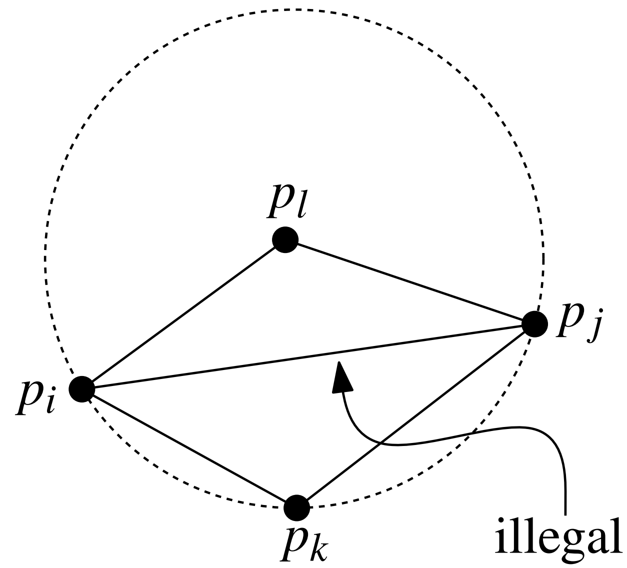
Lemma: The edge $\overline{p_i p_j}$ is illegal if and only if p_l lies in the interior of the circle C.

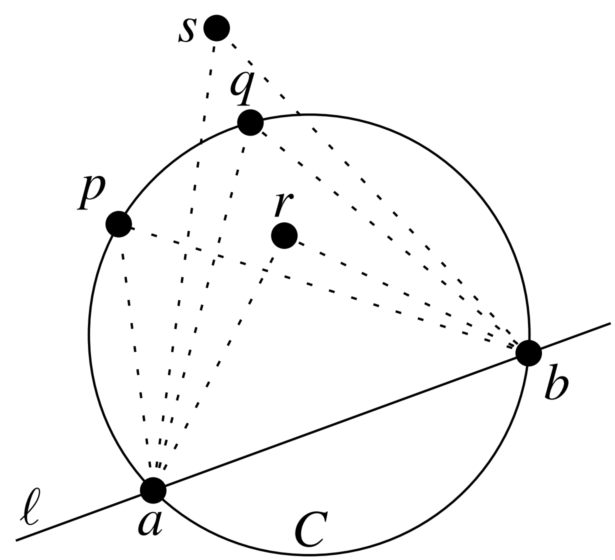


Thales Theorem

Theorem: Let C be a circle, ℓ a line intersecting C in points a and b, and p,q,r,s points lying on the same side of ℓ. Suppose that p,q lie on C, r lies inside C, and s lies outside C. Then

$$\measuredangle arb > \measuredangle apb = \measuredangle aqb > \measuredangle asb,$$

where $\measuredangle abc$ denotes the smaller angle defined by three points a,b,c.



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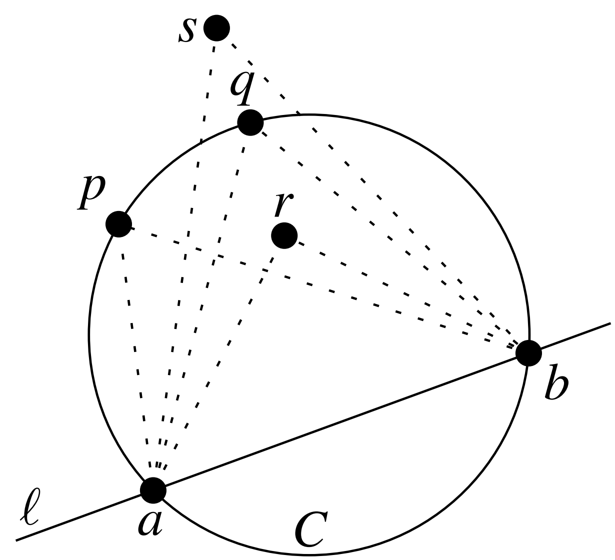

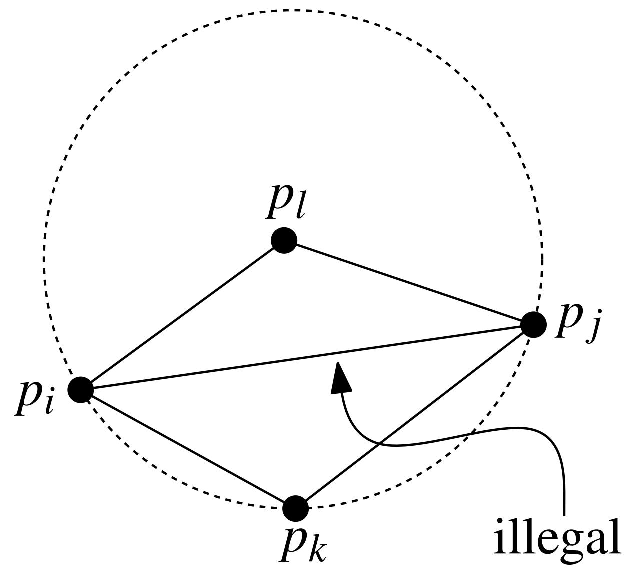

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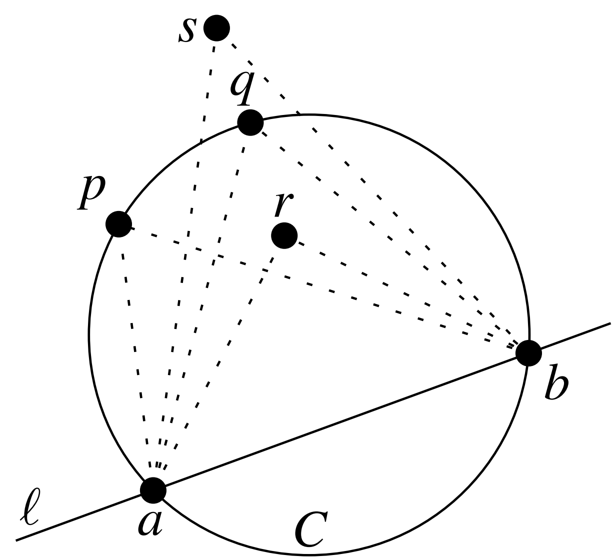

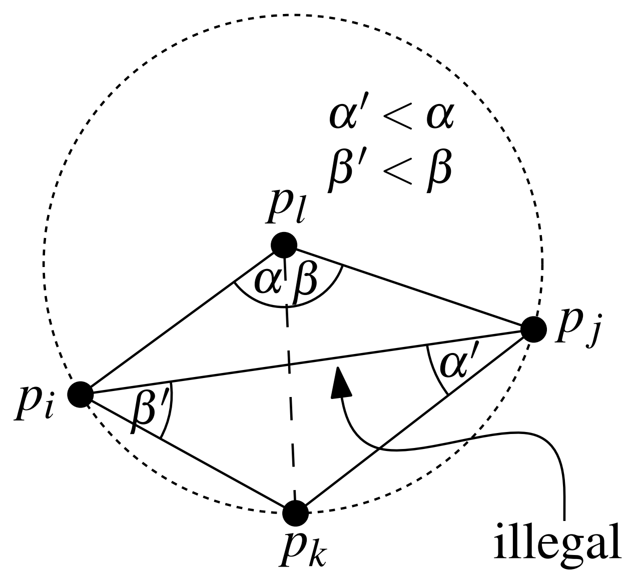

Legal Triangulations

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Algorithm LegalTriangulation($\mathcal{T}$)
Input. A triangulation $\mathcal{T}$ of a point set P.
Output. A legal triangulation of P.
1. **while** $\mathcal{T}$ contains an illegal edge $\overline{p_i p_j}$
2. **do** ($*$ Flip $\overline{p_i p_j}$ $*$)
3. Let $p_i p_j p_k$ and $p_i p_j p_l$ be the two triangles adjacent to $\overline{p_i p_j}$.
4. Remove $\overline{p_i p_j}$ from $\mathcal{T}$, and add $\overline{p_k p_l}$ instead.
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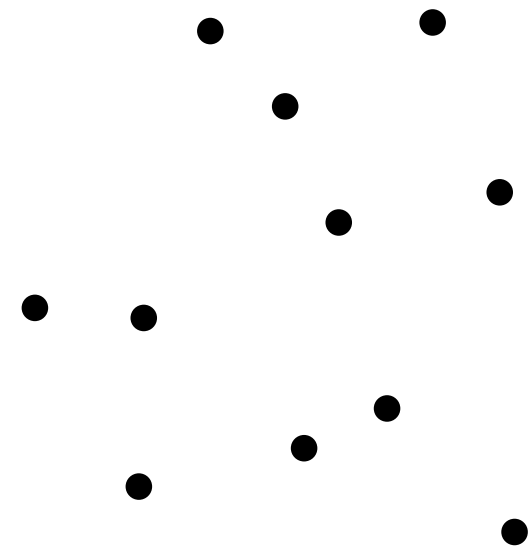
Question: Why does this algorithm terminate?

Voronoi Diagram and Delaunay Graph

- Let P be a set of n points in the plane.
- The Voronoi diagram $\text{Vor}(P)$ is the subdivision of the plane into Voronoi cells $\mathcal{V}(p)$ for all $p \in P$.
- Let $\mathcal{G}$ be the *dual graph* of $\text{Vor}(P)$.
- The Delaunay graph $\mathcal{DG}(P)$ is the *straight line embedding* of $\mathcal{G}$.
- **Question:** How can we compute the Delaunay graph?



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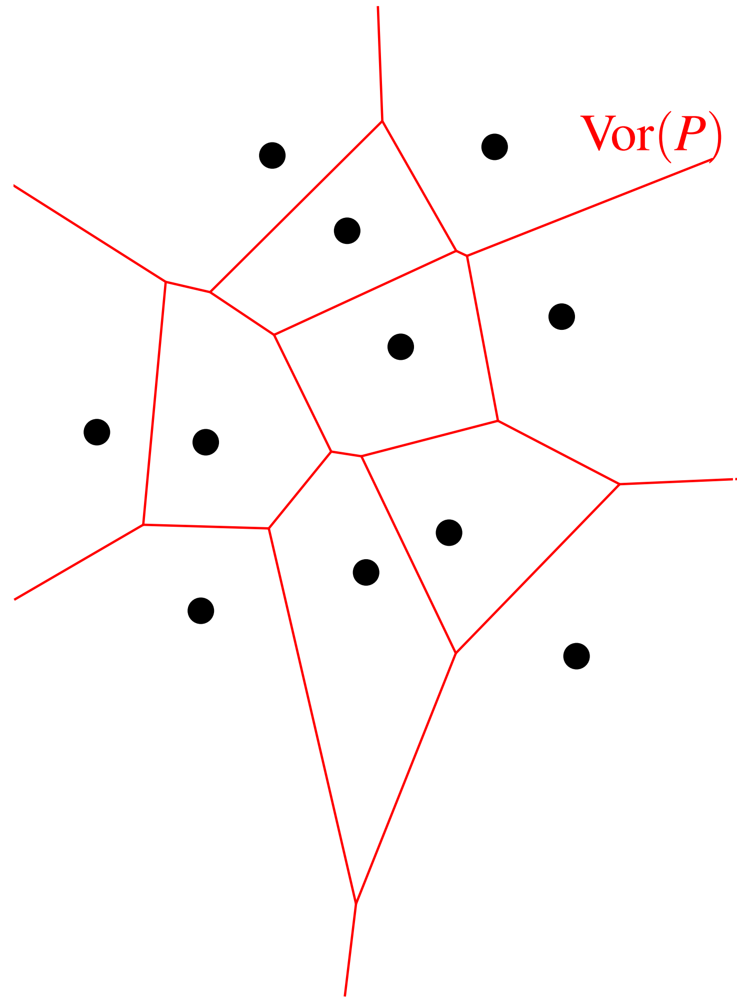

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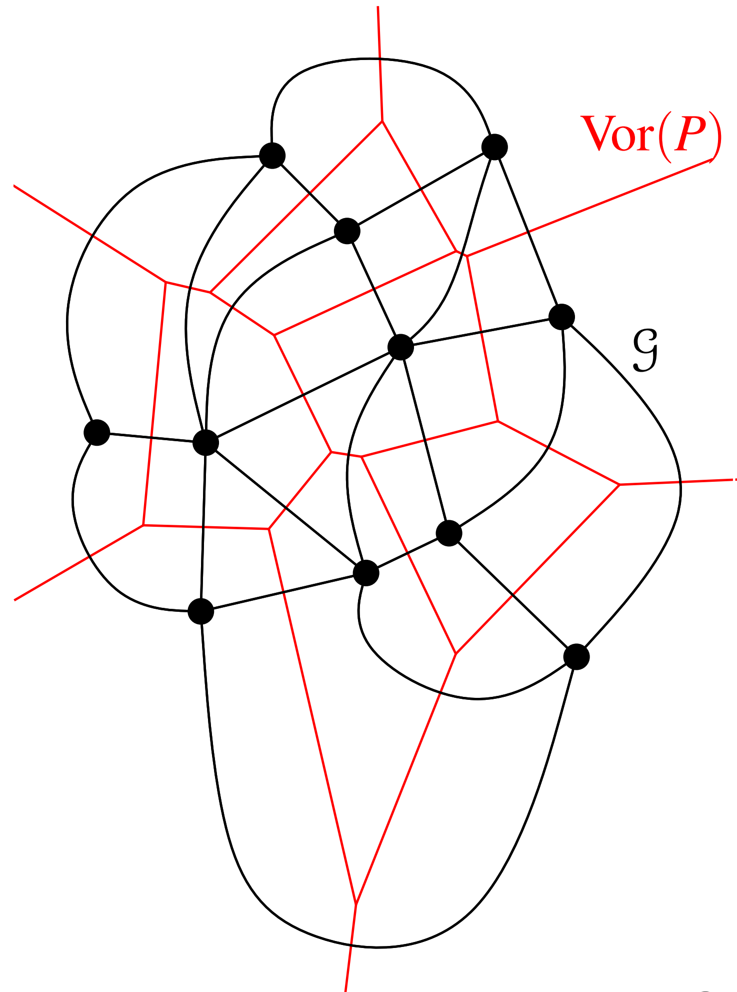

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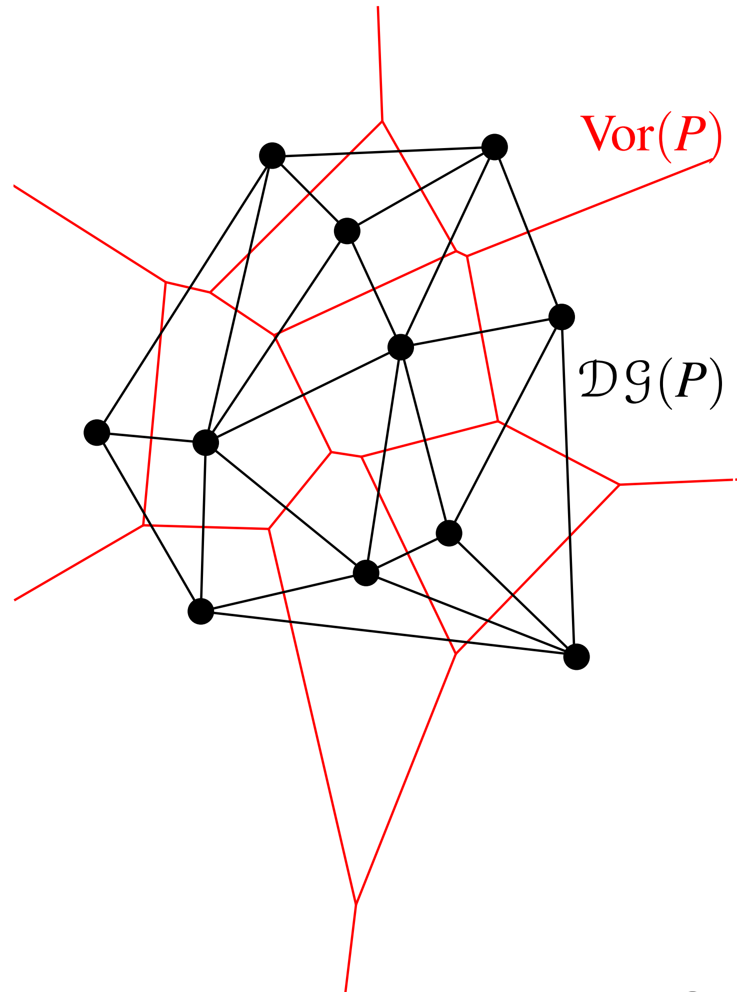

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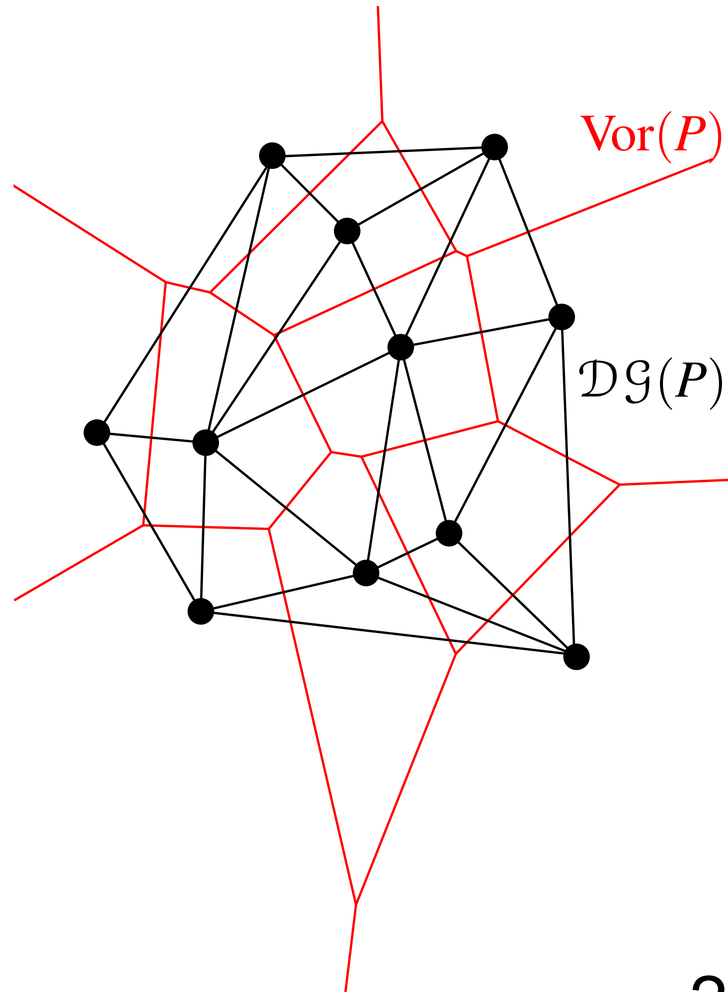
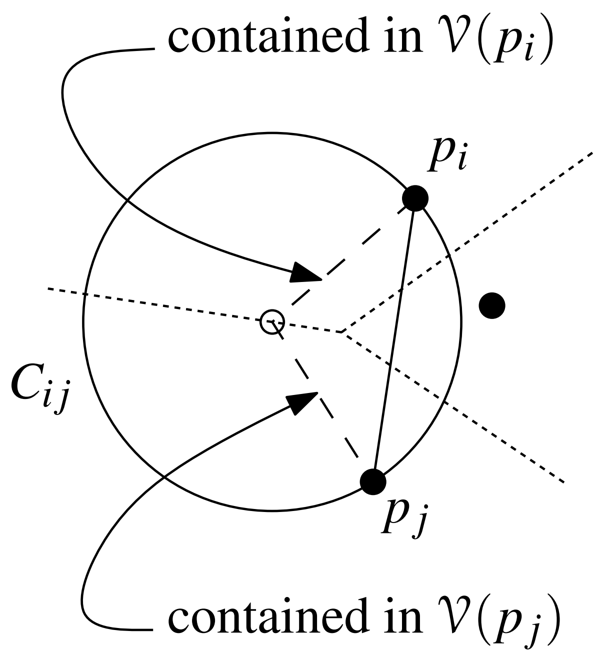
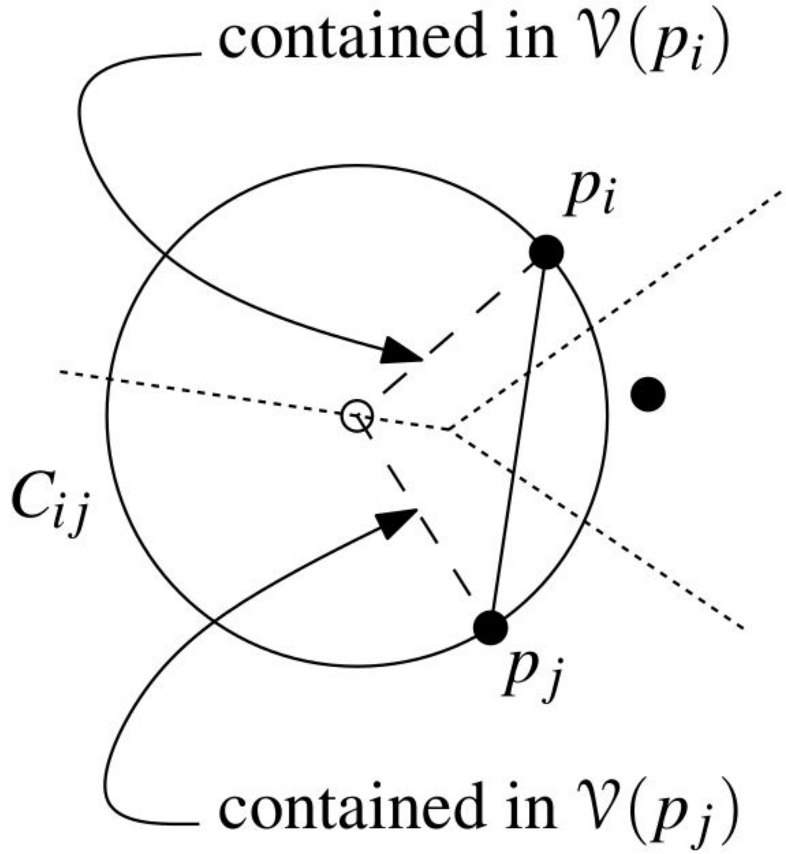


Planarity of the Delaunay Graph

Theorem: The Delaunay graph of a planar point set is a plane graph.





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p_i
C_{ij}
p_j
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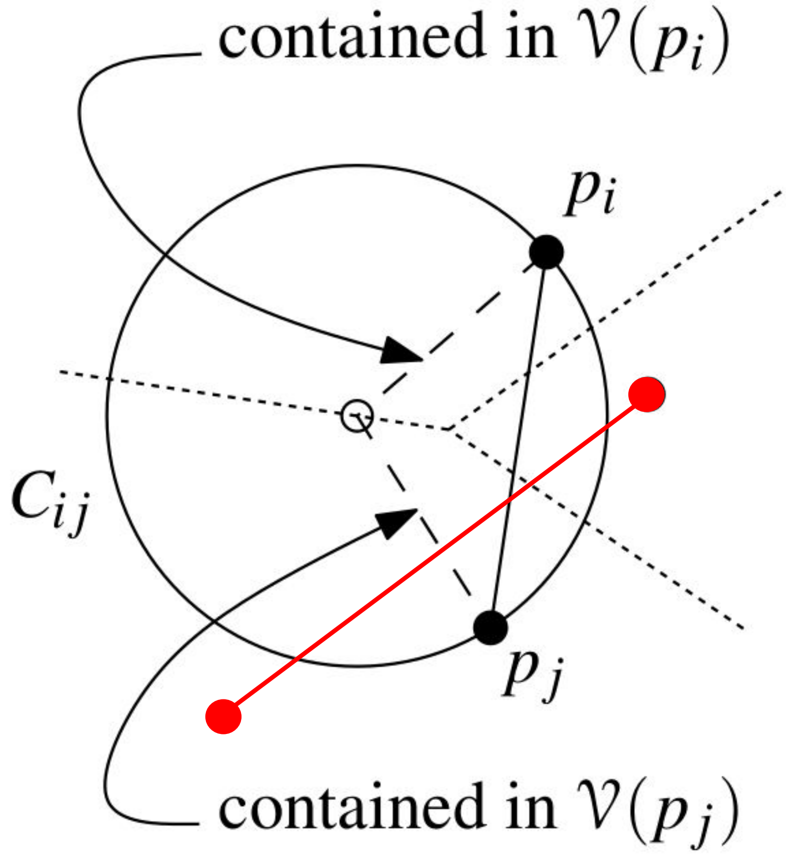
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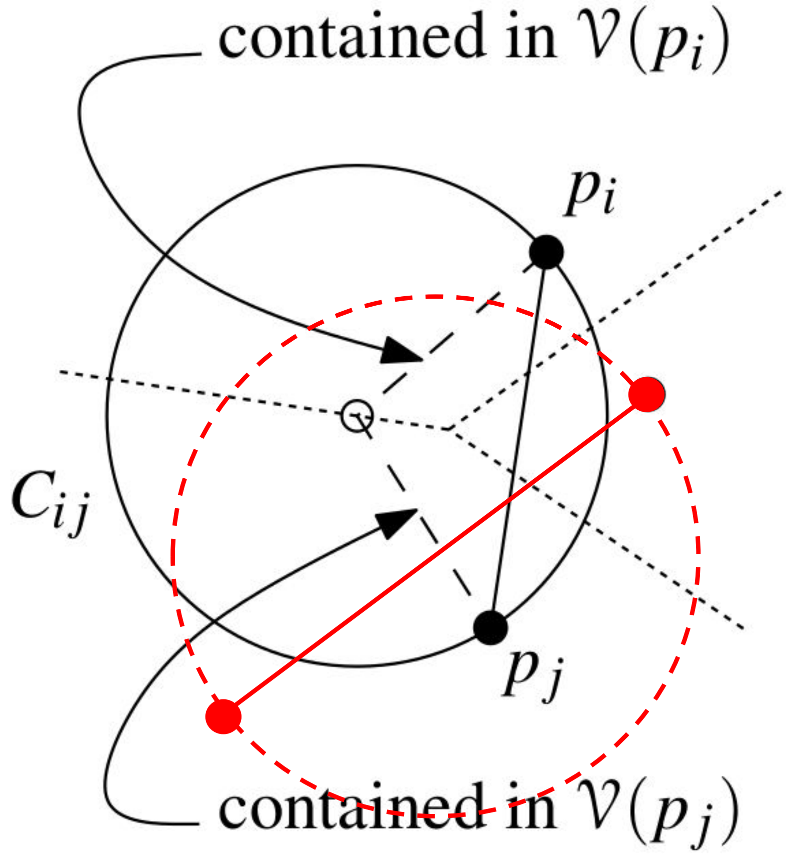
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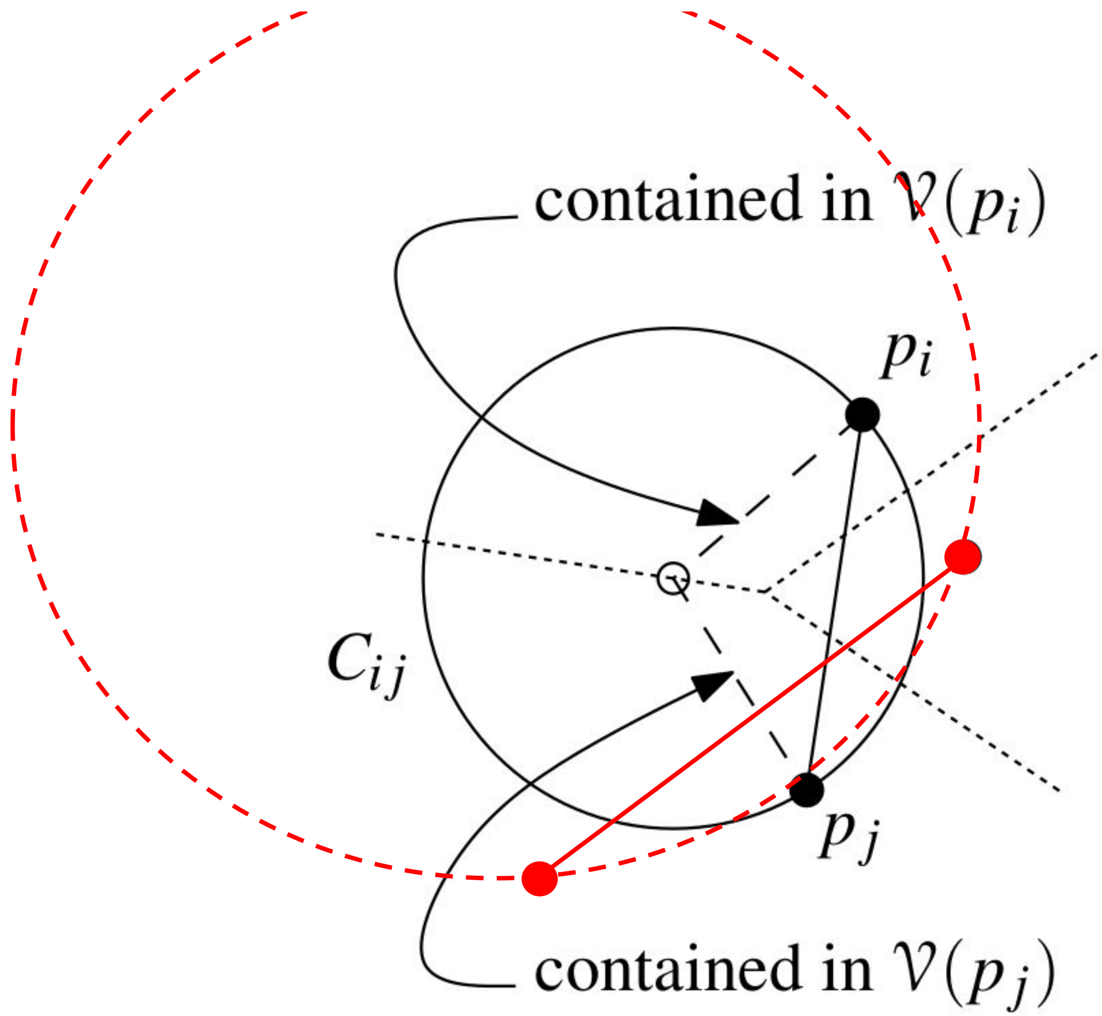

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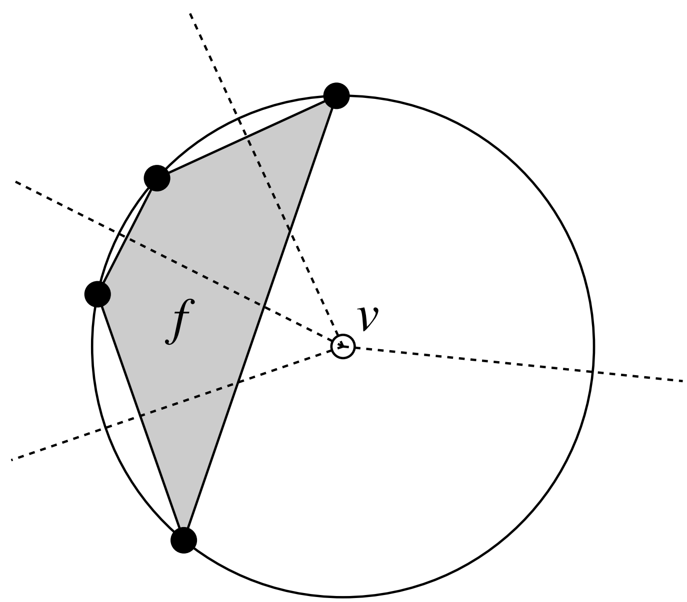
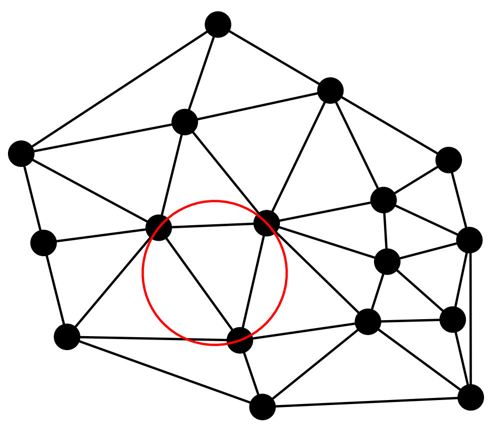
Delaunay Triangulation

If the point set P is in *general position* then the Delaunay graph is a triangulation.



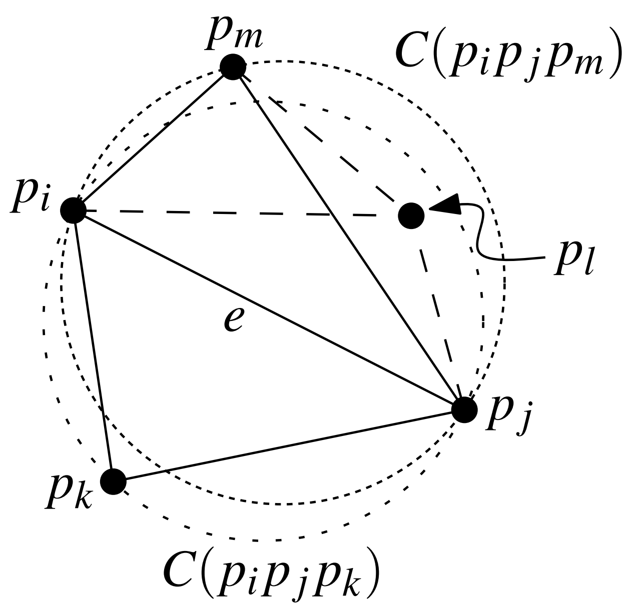
Empty Circle Property

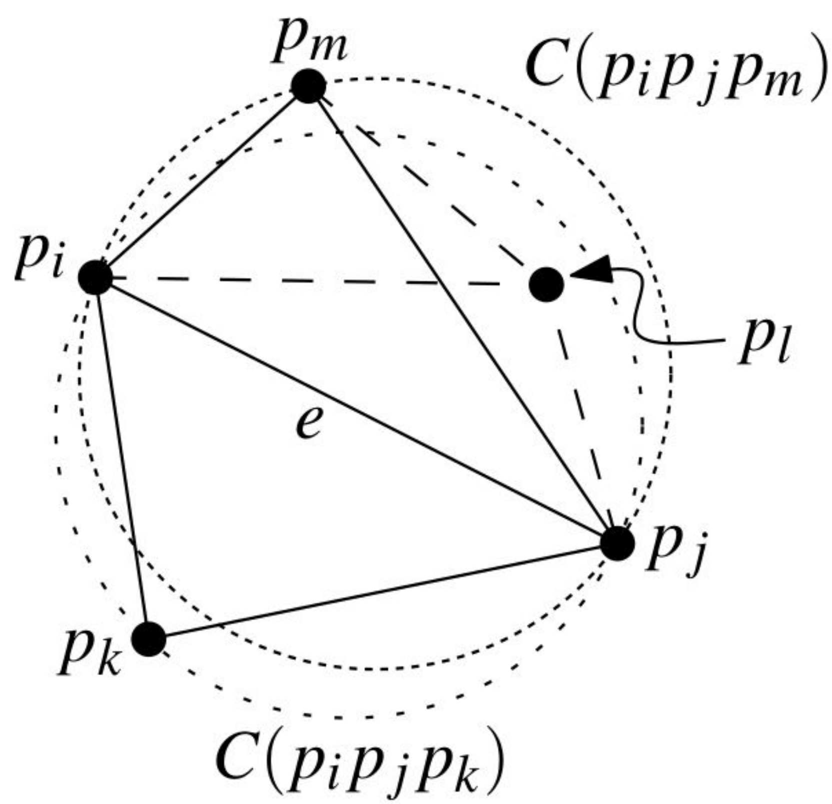
Theorem: Let P be a set of points in the plane, and let $\mathcal{T}$ be a triangulation of P. Then $\mathcal{T}$ is a Delaunay triangulation of P if and only if the circumcircle of any triangle of $\mathcal{T}$ does not contain a point of P in its interior.



Delaunay Triangulations and Legal Triangulations

Theorem: Let P be a set of points in the plane. A triangulation $\mathcal{T}$ of P is legal if and only if $\mathcal{T}$ is a Delaunay triangulation.





p_m
$C(p_i p_j p_m)$
p_i
p_l
e
p_j
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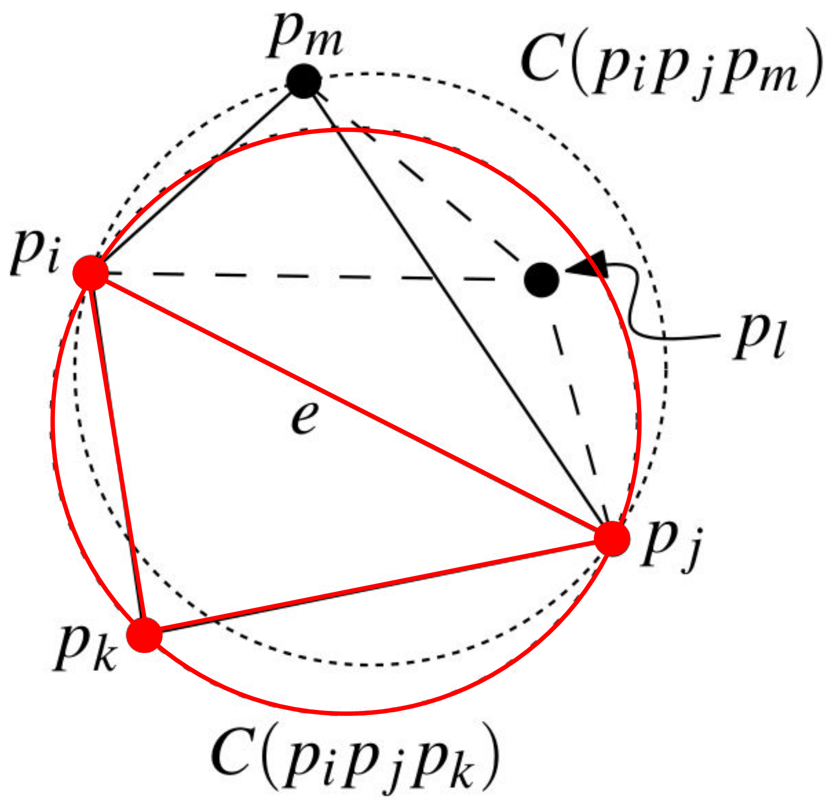

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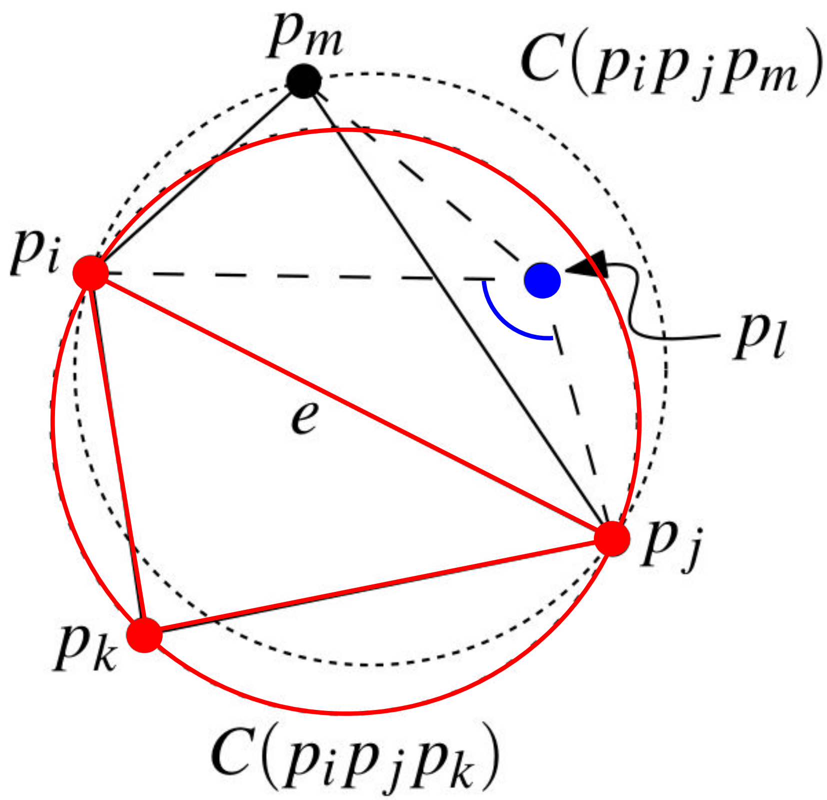
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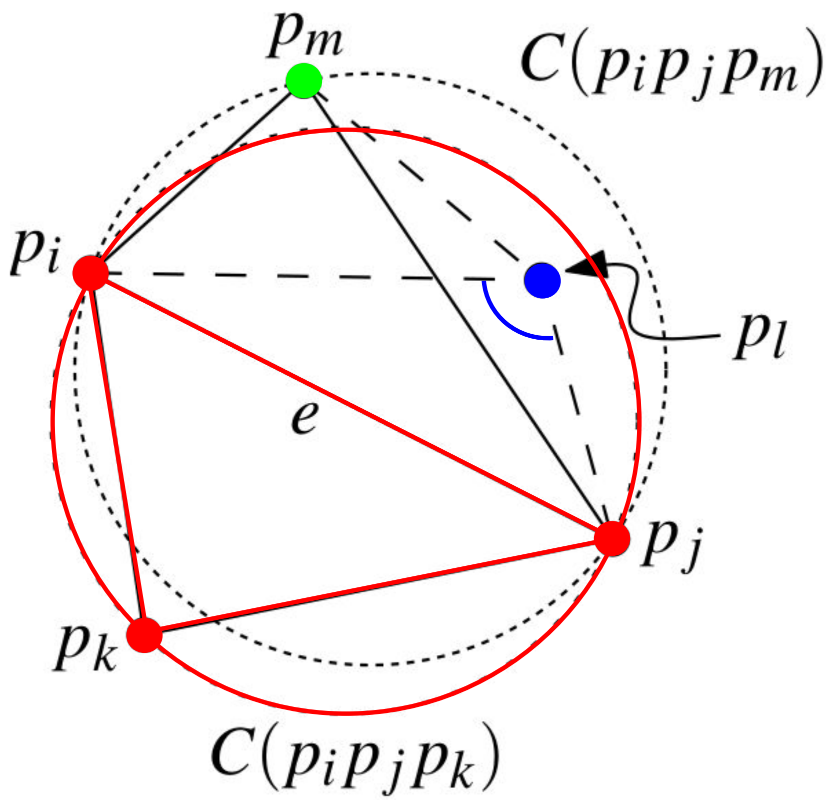

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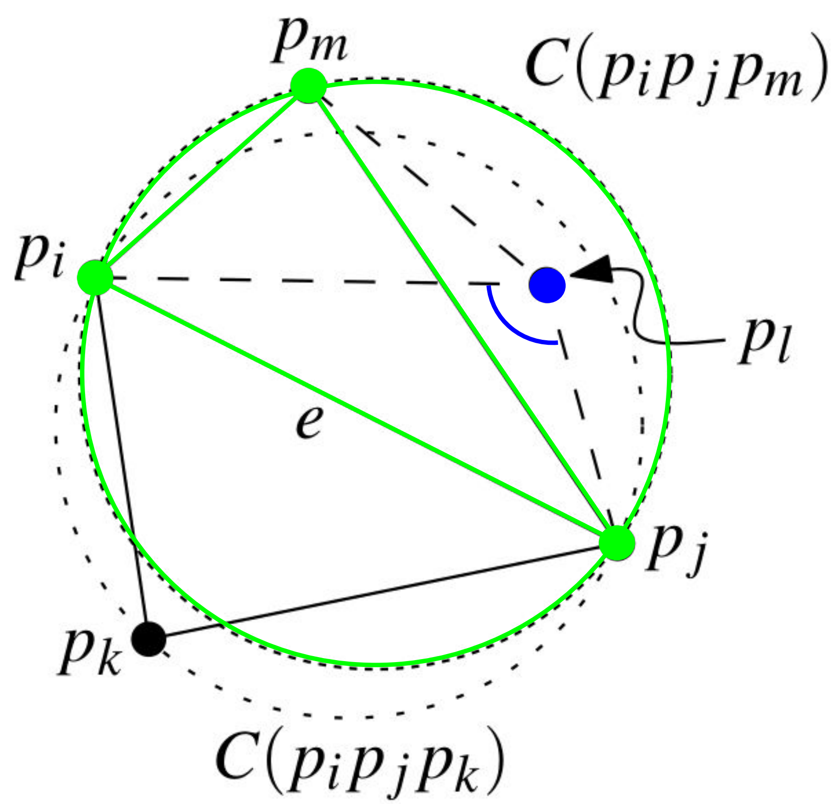
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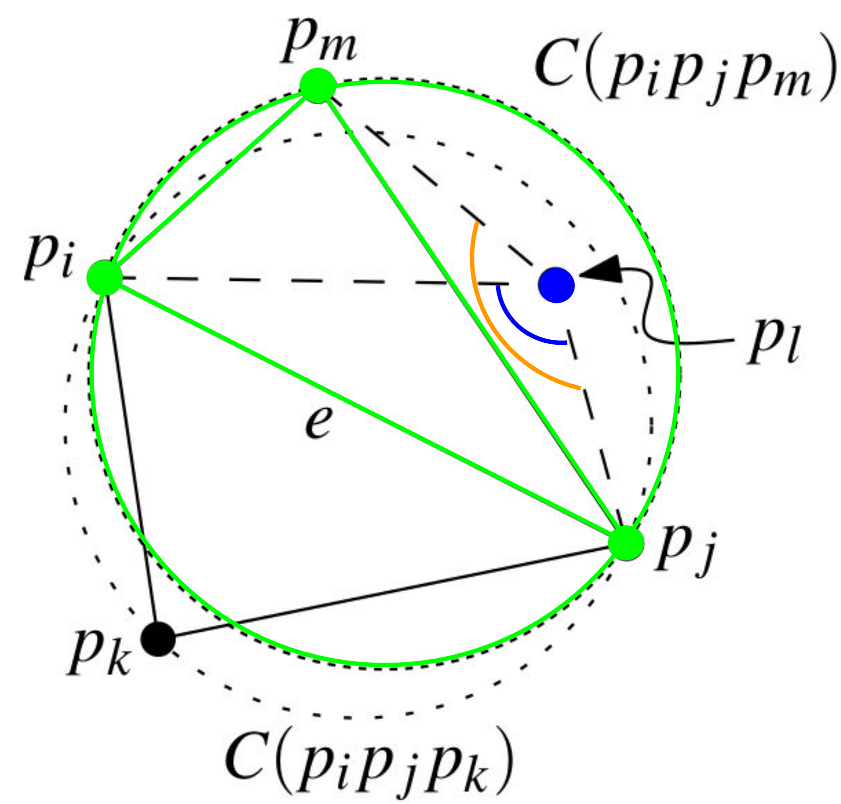

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p_j
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$C(p_i p_j p_k)$

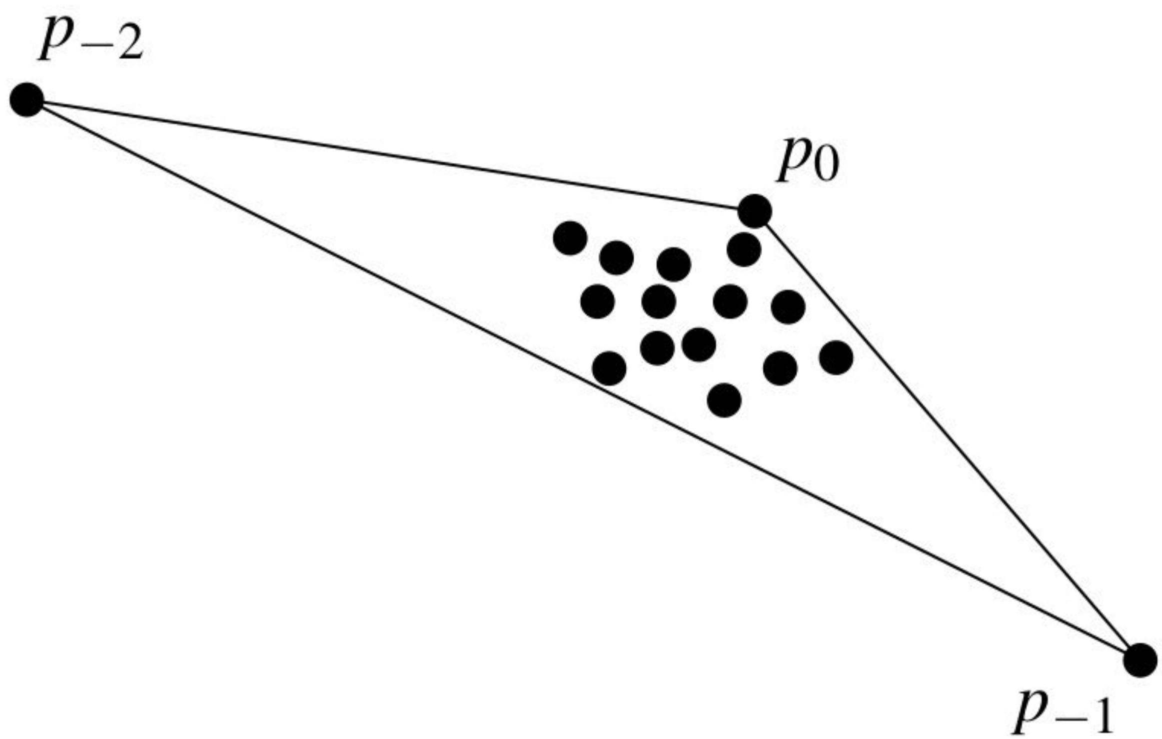
Angle Optimality and Delaunay Triangulations

Theorem: Let P be a set of points in the plane. Any angle-optimal triangulation of P is a Delaunay triangulation of P. Furthermore, any Delaunay triangulation of P maximizes the minimum angle over all triangulations of P.

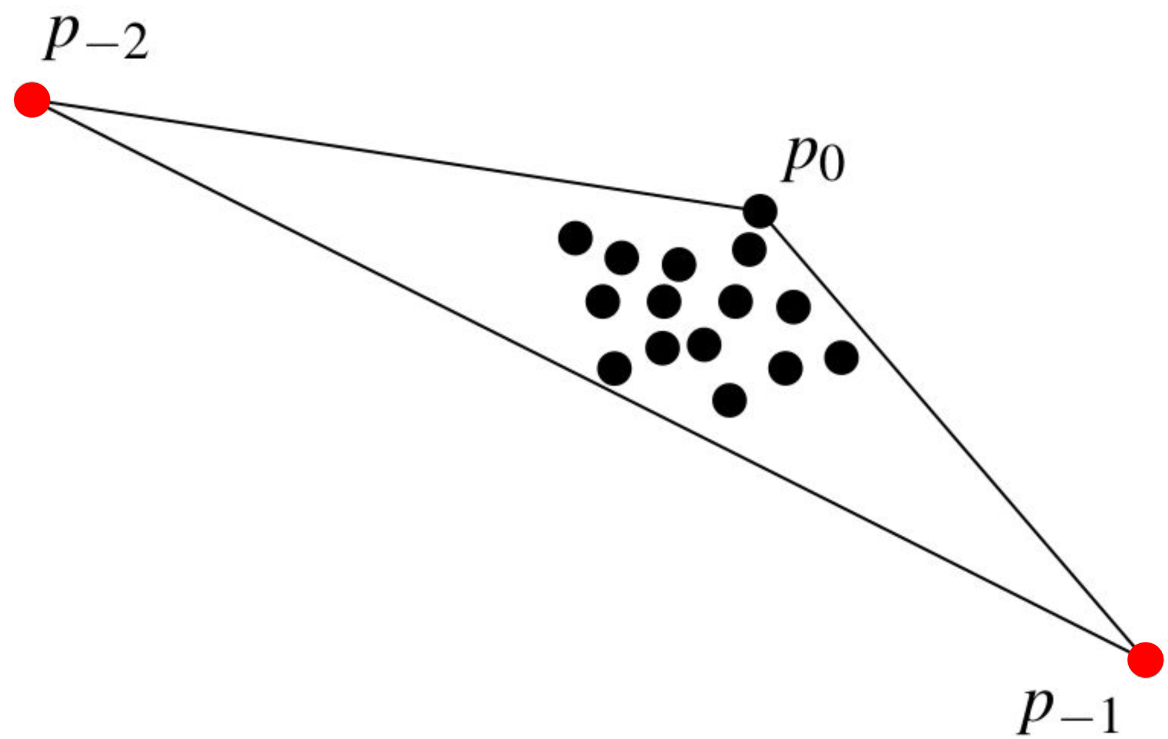
Randomized Incremental Construction

Algorithm DELAUNAYTRIANGULATION(P)
Input. A set P of $n+1$ points in the plane.
Output. A Delaunay triangulation of P.

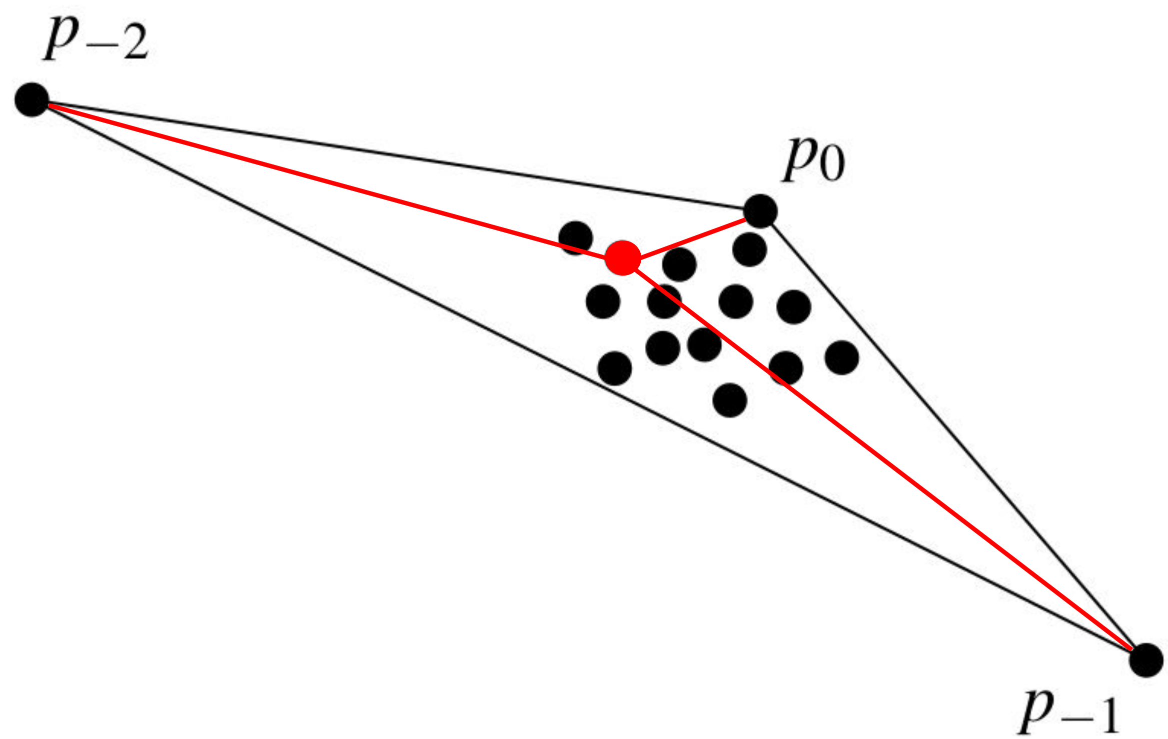
1. Initialize $\mathcal{T}$ as the triangulation consisting of an outer triangle $p_0 p_{-1} p_{-2}$ containing points of P, where p_0 is the lexicographically highest point of P.
2. Compute a random permutation $p_1, p_2, \ldots, p_n$ of $P \setminus \{p_0\}$.
3. **for** $r \leftarrow 1$ **to** n
4. **do**
5. LOCATE($p_r, \mathcal{T}$)
6. INSERT($p_r, \mathcal{T}$)
7. Discard p_{-1} and p_{-2} with all their incident edges from $\mathcal{T}$.
8. **return** $\mathcal{T}$



p_{-2}
p_0
p_{-1}



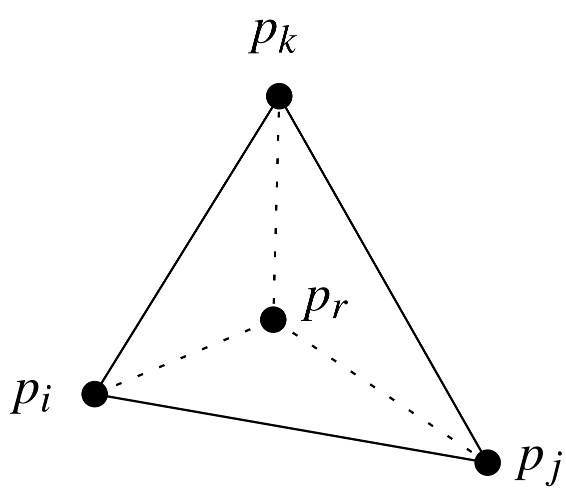
p_{-2}
p_0
p_{-1}



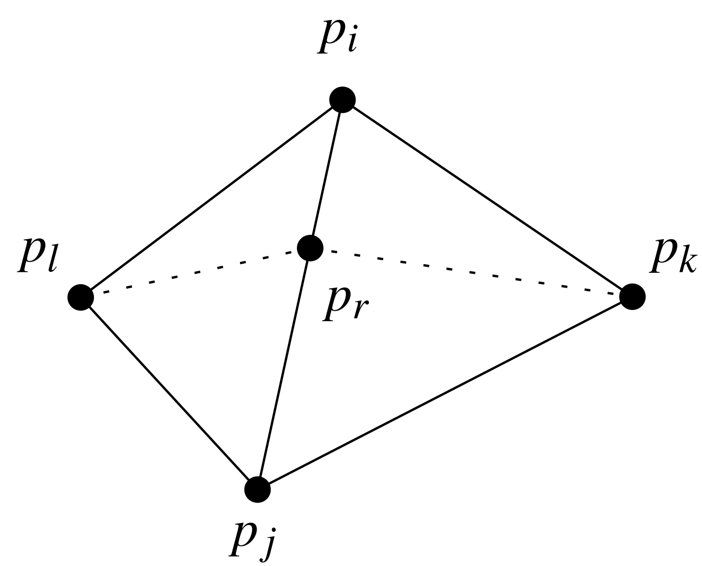
p_{-2}
p_0
p_{-1}

Randomized Incremental Construction

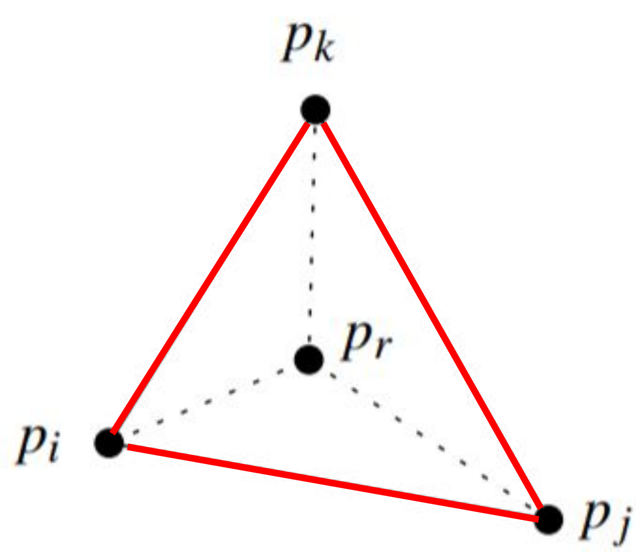
p_r lies in the interior of a triangle



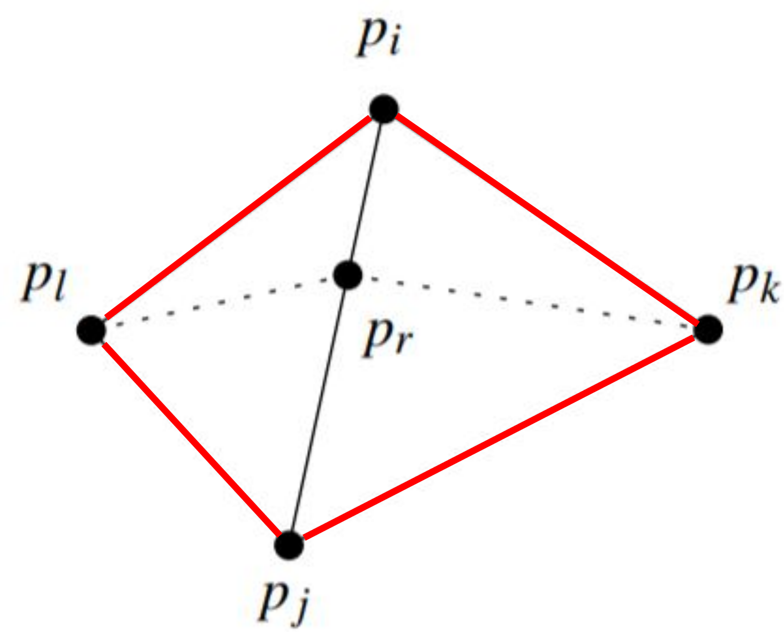
p_r falls on an edge



p_r lies in the interior of a triangle



p_r falls on an edge



Randomized Incremental Construction

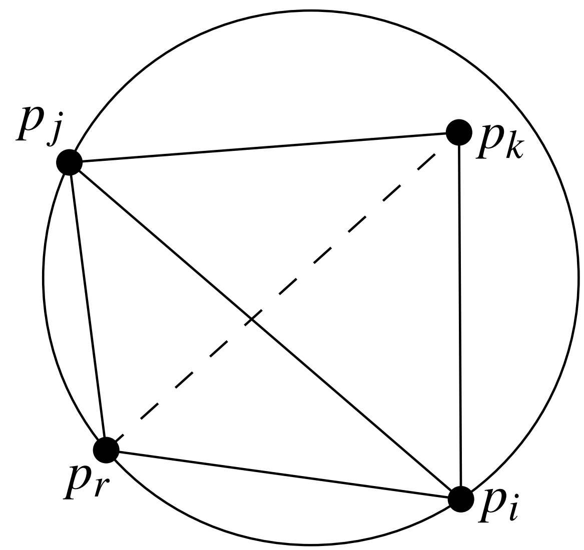
INSERT($p_r, \mathcal{T}$)

1. **if** p_r lies in the interior of the triangle $p_ip_jp_k$
2. **then** Add edges from p_r to the three vertices of $p_ip_jp_k$, thereby splitting $p_ip_jp_k$ into three triangles.
3. LEGALIZEEDGE($p_r, \overline{p_ip_j}, \mathcal{T}$)
4. LEGALIZEEDGE($p_r, \overline{p_jp_k}, \mathcal{T}$)
5. LEGALIZEEDGE($p_r, \overline{p_kp_i}, \mathcal{T}$)
6. **else** ($*$ p_r lies on an edge of $p_ip_jp_k$, say the edge $\overline{p_ip_j}$ $*$)
7. Add edges from p_r to p_k and to the third vertex p_l of the other triangle that is incident to $\overline{p_ip_j}$, thereby splitting the two triangles incident to $\overline{p_ip_j}$ into four triangles.
8. LEGALIZEEDGE($p_r, \overline{p_ip_l}, \mathcal{T}$)
9. LEGALIZEEDGE($p_r, \overline{p_lp_j}, \mathcal{T}$)
10. LEGALIZEEDGE($p_r, \overline{p_jp_k}, \mathcal{T}$)
11. LEGALIZEEDGE($p_r, \overline{p_kp_i}, \mathcal{T}$)

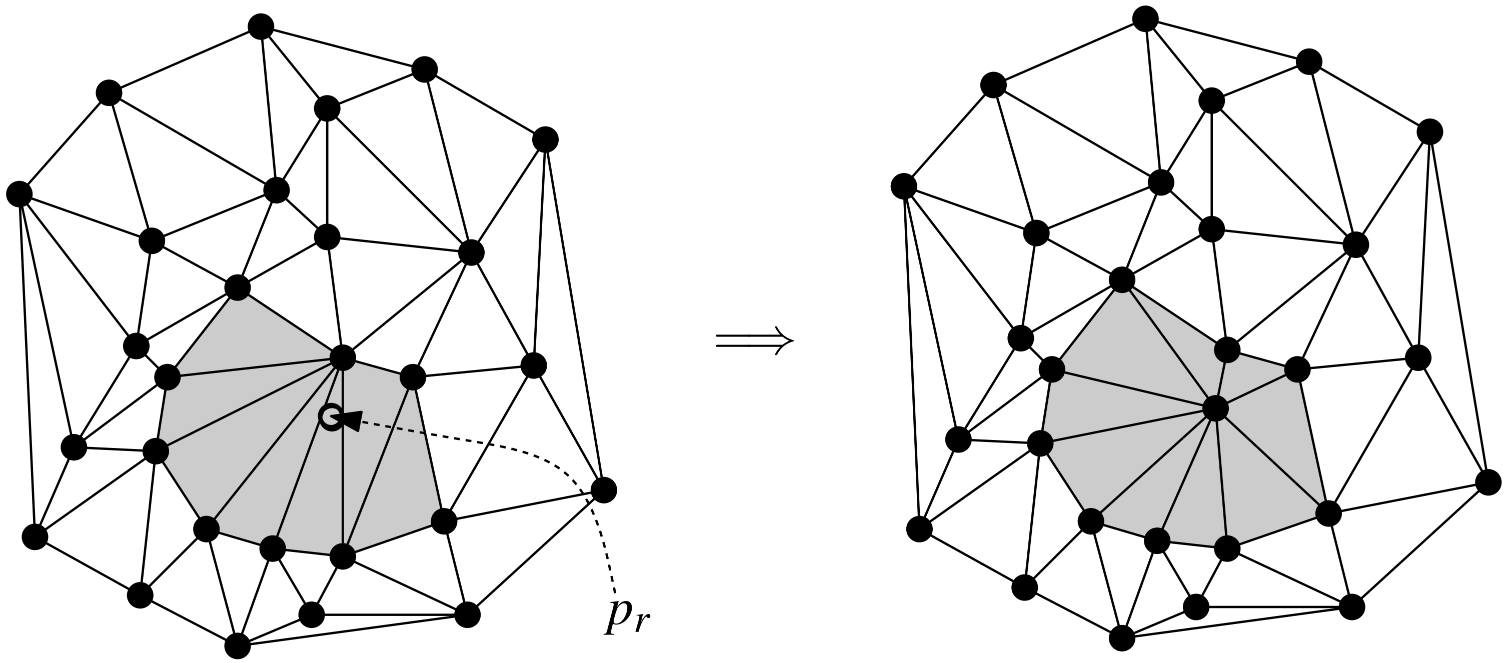
Randomized Incremental Construction

$\textsc{LegalizeEdge}(p_r, \overline{p_i p_j}, \mathcal{T})$

1. (∗ The point being inserted is p_r, and $\overline{p_i p_j}$ is the edge of $\mathcal{T}$ that may need to be flipped. ∗)
2. **if** $\overline{p_i p_j}$ is illegal
3. **then** Let $p_i p_j p_k$ be the triangle adjacent to $p_r p_i p_j$ along $\overline{p_i p_j}$.
4. (∗ Flip $\overline{p_i p_j}$: ∗) Replace $\overline{p_i p_j}$ with $\overline{p_r p_k}$.
5. $\textsc{LegalizeEdge}(p_r, \overline{p_i p_k}, \mathcal{T})$
6. $\textsc{LegalizeEdge}(p_r, \overline{p_k p_j}, \mathcal{T})$

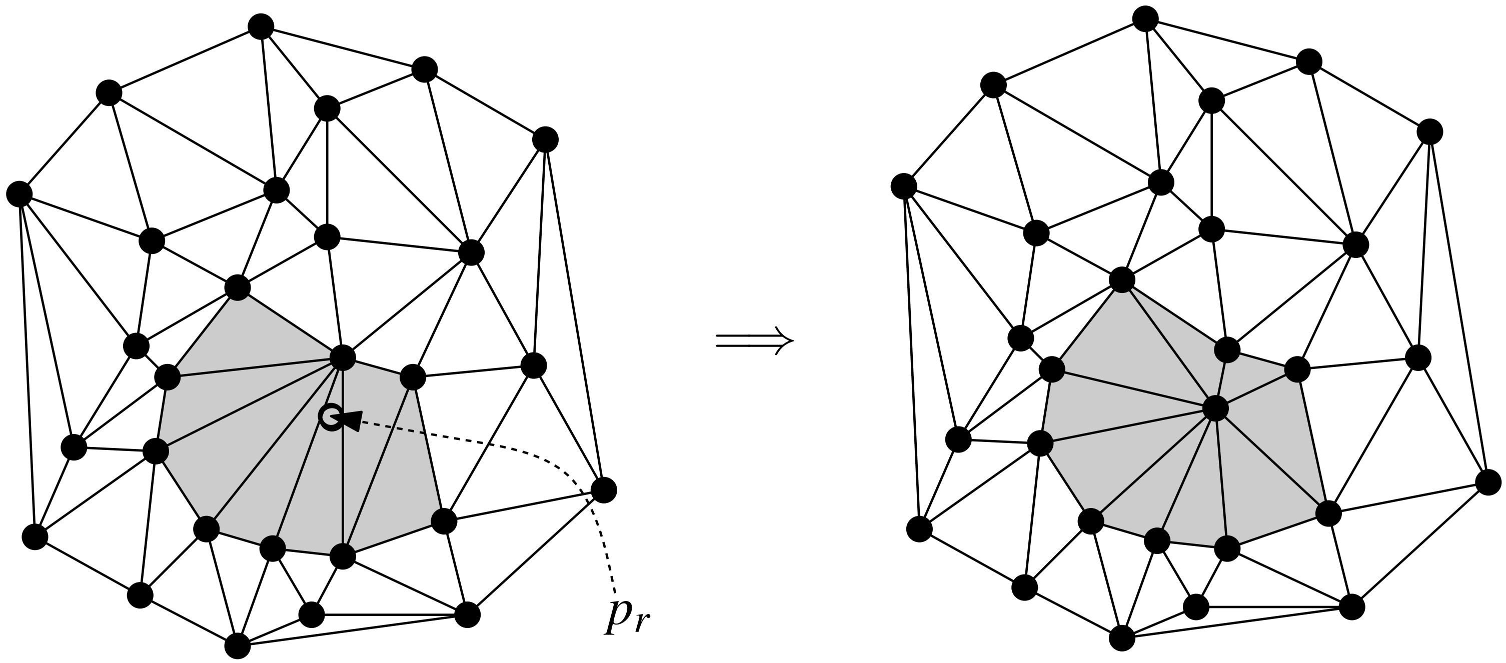


Randomized Incremental Construction



All edges created are incident to p_r.

Randomized Incremental Construction



All edges created are incident to p_r.

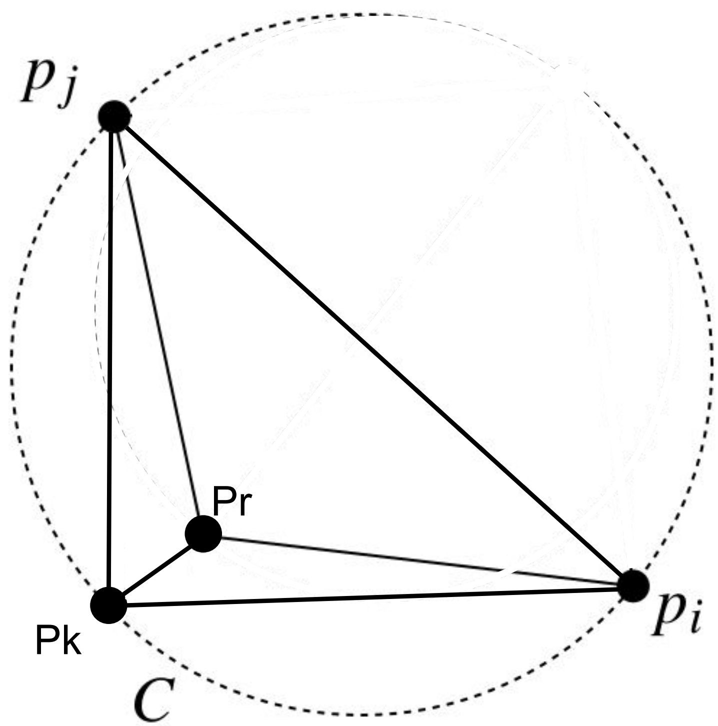
Correctness: Are new edges legal?

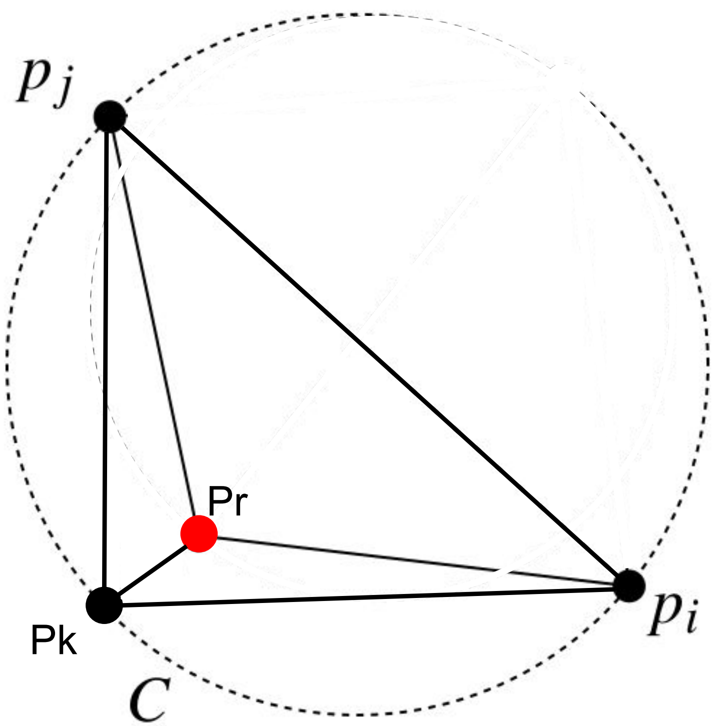
Correctness

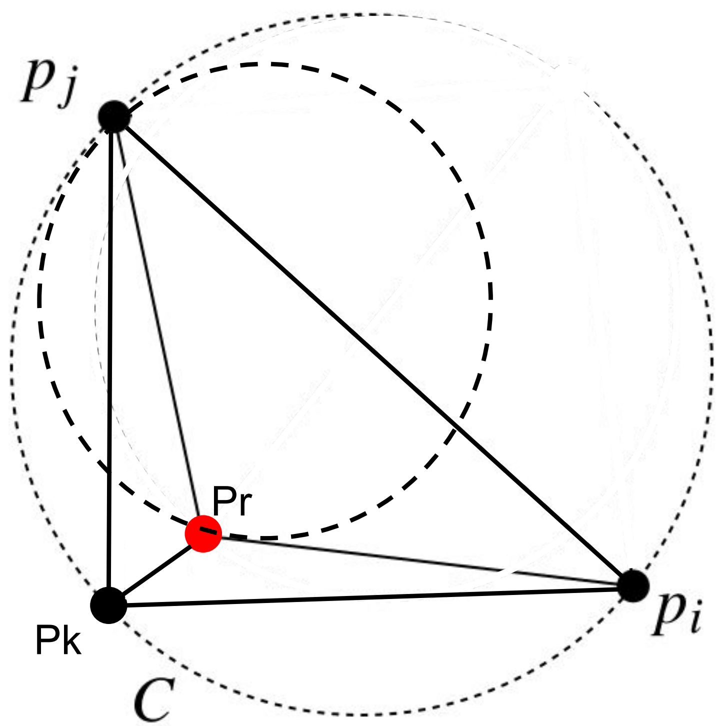
- Newly added edges
- Edges due to flipping

Correctness

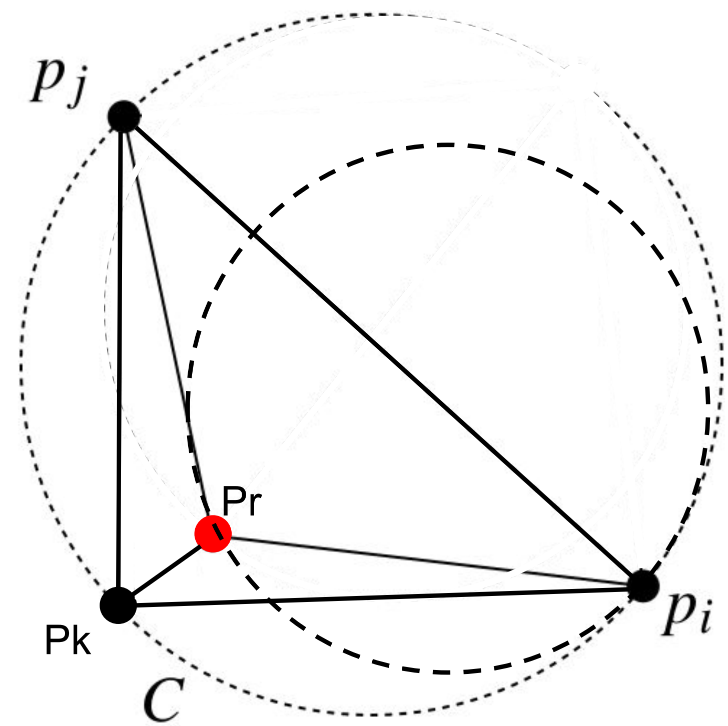
- **Newly added edges**
- Edges due to flipping


p_j
Pr
Pk
p_i
C

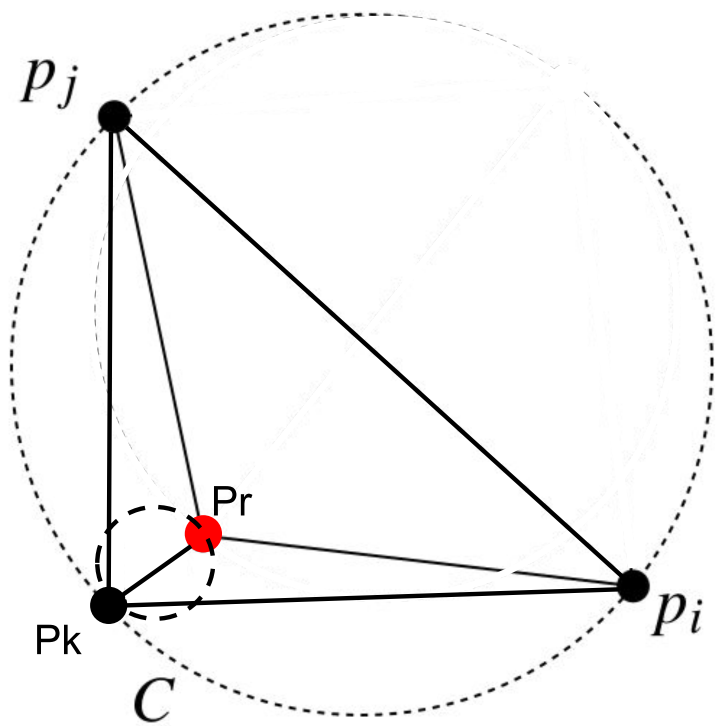

p_j
Pr
Pk
p_i
C



p_j
Pr
Pk
p_i
C

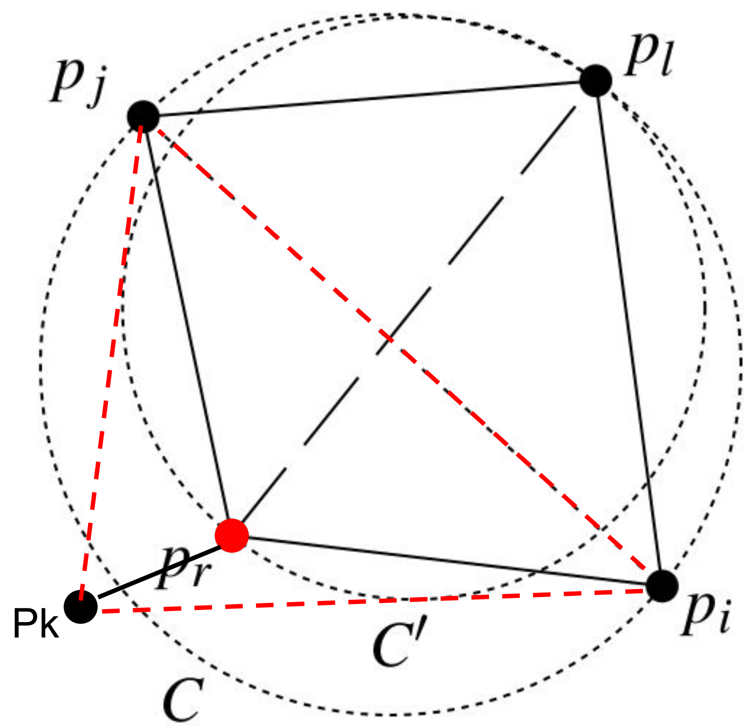


p_j
Pr
Pk
p_i
C

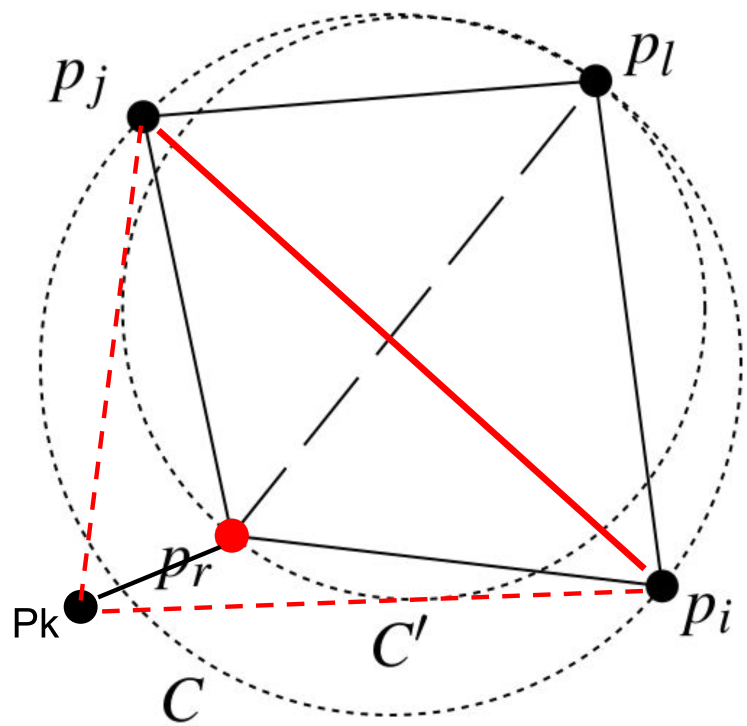

p_j
Pr
Pk
p_i
C

Correctness

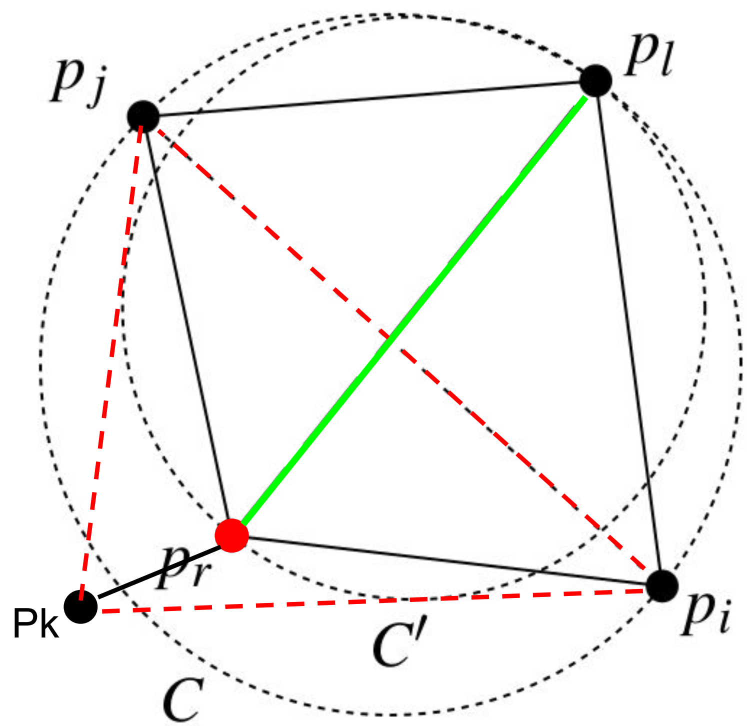
- Newly added edges
- **Edges due to flipping**



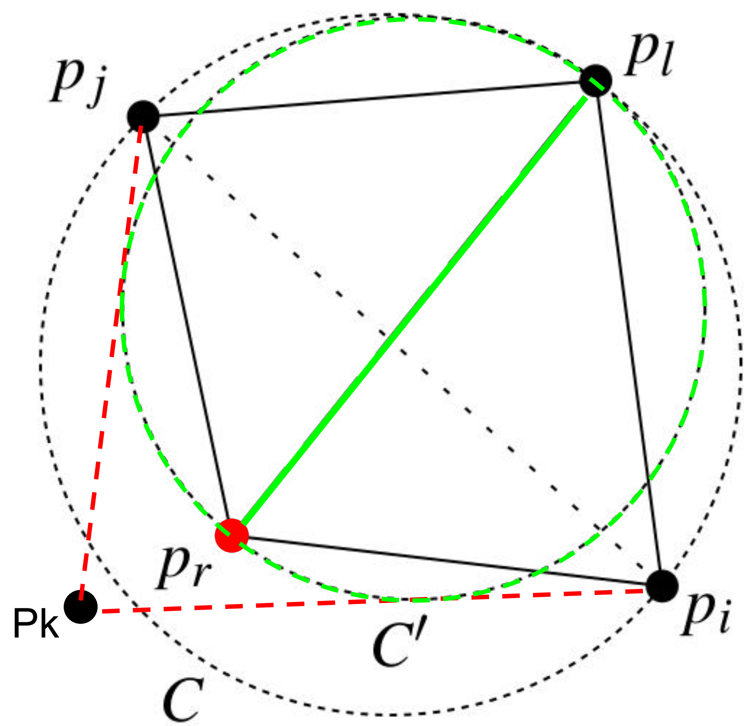
p_j
p_l
p_r
Pk
p_i
C'
C



p_j
p_l
p_r
Pk
p_i
C'
C



p_j
p_l
p_r
Pk
p_i
C'
C



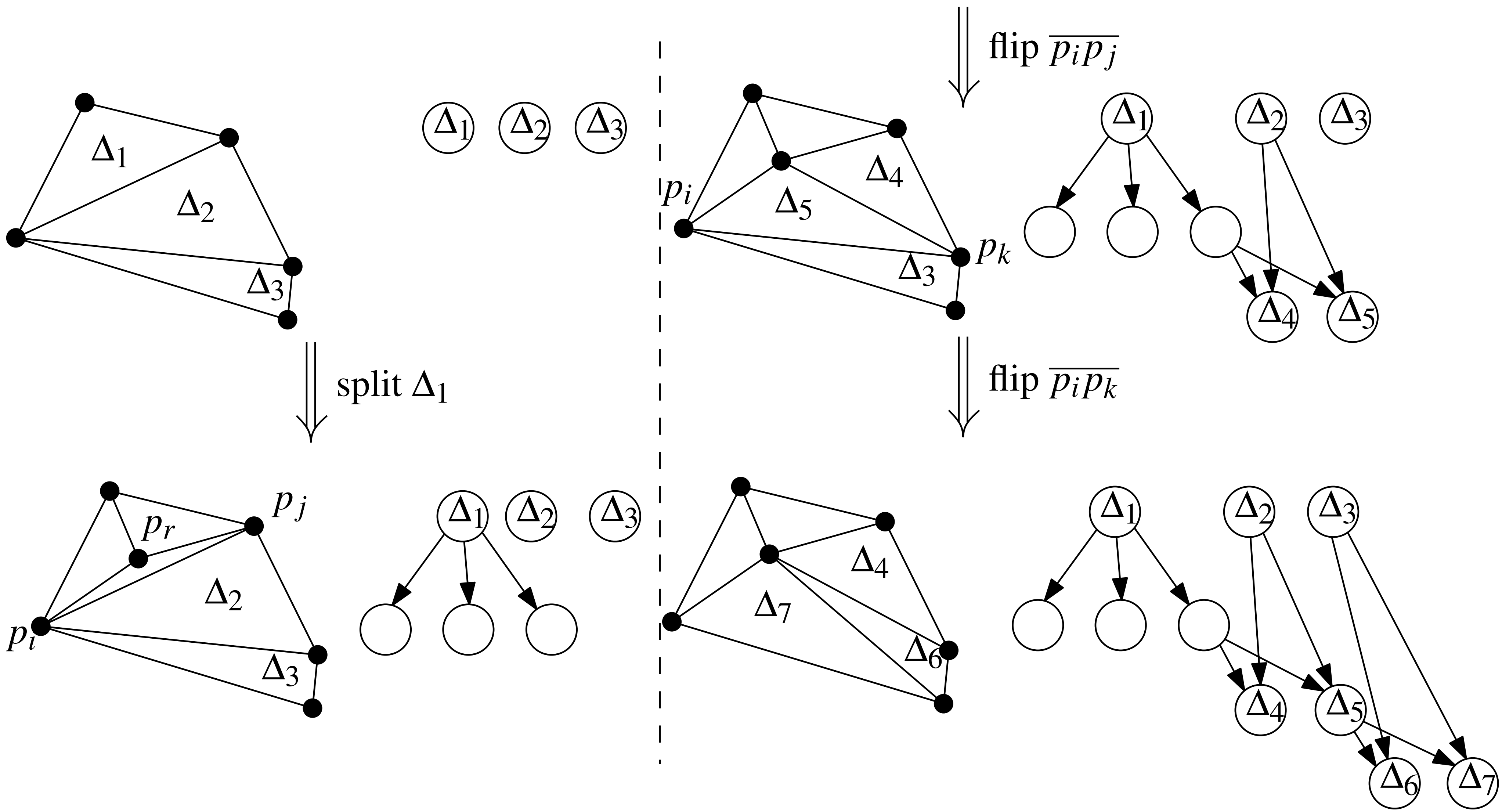
p_j
p_l
p_r
p_i
Pk
C'
C

Randomized Incremental Construction

Initializing triangulation: treat p_{-1} and p_{-2} symbolically.
No actual coordinates.
Modify tests for point location and illegal edges to work as if far away.

Point location: search data structure.
Point visits triangles of previous triangulations that contain it.

Randomized Incremental Construction



Analysis

- Expected total number of triangles created is O(n)
  - Space usage (point data structure) O(n)
  - Expected time other than point location queries O(n)
- Expected total number of triangles visited while search point location data structure is O(n log n)

Analysis

Lemma: Total number of triangles created is at most 9n + 1

- For each of p_r added, let it have k incident edges
- It creates at most 2k - 3 triangles
- Known: delaunay graph has at most 3(r+3)-6 edges, three of which is the outer triangle
- 2[3(r+3) - 9] = 6r -- expected degree of a random point is 6
- 2 * 6 - 3 = 9
- +1 for the outer triangle