The Input/Output Complexity of Sorting and Related Problems

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6.886

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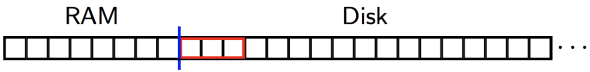
Overview

- I/O Model
- Tight bounds for worst and average case for the following problems
    1. Sorting
    2. Permuting
    3. FFT/permutation networks
    4. Matrix transposition
- Analysis of I/O bounds for algorithms

I/O Model

Memory model:



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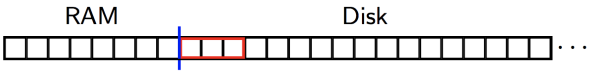

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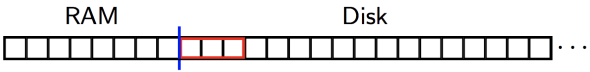

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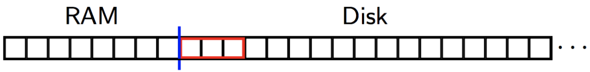

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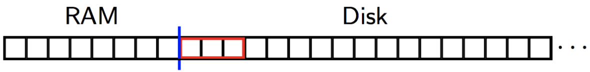

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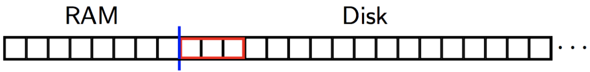

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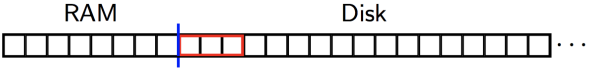

- Memory is divided into internal memory (holds **M** records) and secondary storage/disk ($>>$ **M**)
- We can think of both together as a single contiguous array, where internal memory goes from $x[1], x[2], \cdots, x[M]$ and secondary storage from $x[M+1], x[M+2], \cdots$

Sorting

- Problem: The internal memory is empty, and the N records reside at the beginning of the disk
- Goal: The internal memory is empty, and the N records reside at the beginning of the disk in sorted nondecreasing order by their key values.
- Some notation: We denote the N records as $R_1, R_2, \cdots, R_N$. At the start of the problem, $x[M+i] = R_i \ \forall 1 \le i \le N$.

Permutation

- Problem: The internal memory is empty, and the N records reside at the beginning of the disk (same as sorting).
- Goal: The internal memory is empty, and the N records reside at the beginning of the disk. The key values of the N records form a permutation of $\{1, 2, \cdots, N\}$.
- What is the relationship between sorting and permuting?

External Merge Sort

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3. Now we begin merging partitions
   3.1 Assume we have P_1 and P_2. We want the get the B first elements in $P_1 \cup P_2$
   3.2 This is clearly contained in $P_1[1:B] \cup P_2[1:B]$.
   3.3 How do we get the next B elements?

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- Approach 1: Reuse the algorithm used for the RAM model. Number of I/Os $O(N)$
- Approach 2: External sort: I/O's $O((N/B)log_{M/B}(N/B))$
- Can we do $O(N/B)$?

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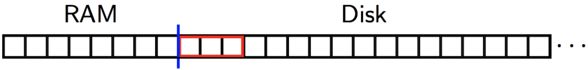

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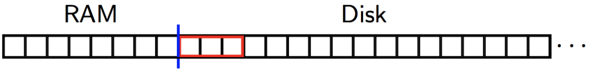

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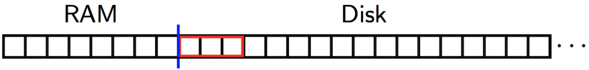

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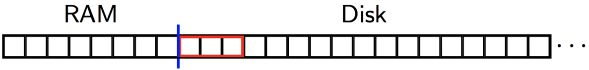

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- Each I/O performed transfers B records that come from the same track.

Main results - Sorting

Theorem

The average and worst case number of I/Os for sorting N records is

$$\theta\left(\frac{N}{PB}\frac{\log(1+N/B)}{\log(1+M/B)}\right)$$

.

- If $M = 2, B = P = 1$ we get the well known $O(N\log(N))$ bound on comparison sort.

Main results - Permutation

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- When M and B are small, we are essentially doing the naive permutation method described before.

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We say a permutation $p_1, p_2, \cdots p_N$ of the N records can be generated at time t if there is some sequence of t I/OS such that after the I/OS all records are in correct permuted order in disk: $x[i] = R_{p_k}$ and $x[j] = R_{p_{k+1}}$ imply $i < j \ \forall i, j, k$.

Sorting

1. Strip out key values and sort in memory.
2. Permute records based off key order.

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Proof Idea: Bound the number of possible permutations that can be generated by t I/Os. Choose smallest t such that the number of possible permutations is $\geq N!$

FFT

Background: Fourier Series

$$\widehat{f}(n) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\theta)\overline{e_n(\theta)}\,d\theta =: \langle f, e_n\rangle. \qquad f(\theta) \sim \sum_{n=-\infty}^{\infty} \widehat{f}(n)e^{in\theta} = \sum_{n=-\infty}^{\infty} \langle f, e_n\rangle\, e_n(\theta).$$

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DFT in Matrix Form:

$$\begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_n & \omega_n^2 & \omega_n^3 & \cdots & \omega_n^{n-1} \\ 1 & \omega_n^2 & \omega_n^4 & \omega_n^6 & \cdots & \omega_n^{2(n-1)} \\ 1 & \omega_n^3 & \omega_n^6 & \omega_n^9 & \cdots & \omega_n^{3(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \omega_n^{3(n-1)} & \cdots & \omega_n^{(n-1)(n-1)} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{n-1} \end{pmatrix}.$$

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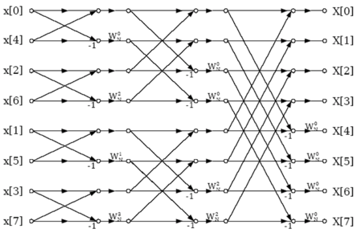
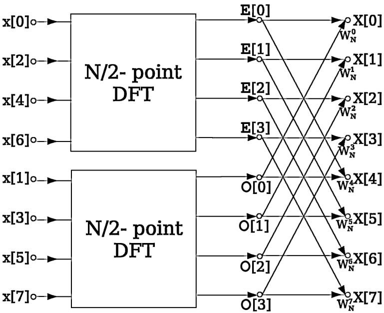
Butterfly Diagram





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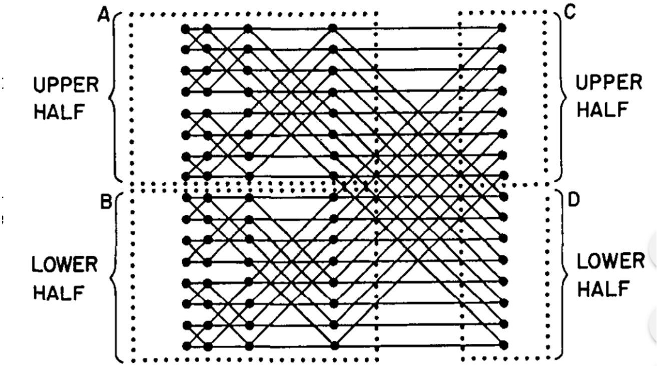

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The average and worst case number of I/Os for for computing the N-input FFT digraph is

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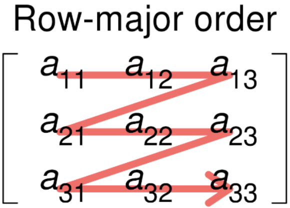

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The number of I/OS required to transpose a p x q matrix stored in row-major order, is

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Compare with theoretical bound:

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6. For $i \in [1, f]$ let p_i be the $i\lceil \frac{N}{f^2} \rceil$ smallest element in G.

Distribution Sort

How do we find our pivots in $O(N/B)$? Inutition: Median of Medians

1. Let $t = N/M$
2. Divide S into t groups: $G_1, \cdots, G_t$, each with M elements.
3. Load each G_i into memory + sort.
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8. Total cost of all k-selections is $O(\frac{N}{fB}f) = O(N/B)$.

Algorithm: Permuting

- Permuting is a special case of sorting.
- Unless B, M is small: then use naive method.

Summary

- Sorting

$$\theta\left(\frac{N}{PB}\frac{\log(1+N/B)}{\log(1+M/B)}\right)$$

- Permuting

$$\theta\left(\min\left(\frac{N}{PB}\frac{\log(1+N/B)}{\log(1+M/B)},\frac{N}{P}\right)\right)$$

- FFT

$$\theta\left(\frac{N}{PB}\frac{\log(1+N/B)}{\log(1+M/B)}\right)$$

- Matrix Transposition

$$\theta\left(\frac{N}{PB}\frac{\log\left(\min 1+N/B, M, 1+\min(p,q)\right)}{\log(1+M/B)}\right)$$