A Functional Approach to External Graph Algorithms [ABW98]

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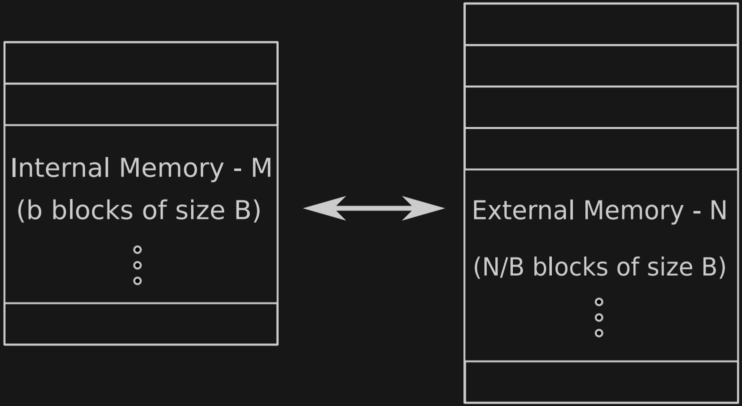
Number of Block Transfers to/from External Memory



- $scan(N) \equiv \mathcal{O}\left(\lceil N/B \rceil\right)$
- $sort(N) \equiv \mathcal{O}\left(scan(N) \log_b \frac{N}{B}\right)$

Functional I/O model for Graph Algorithms

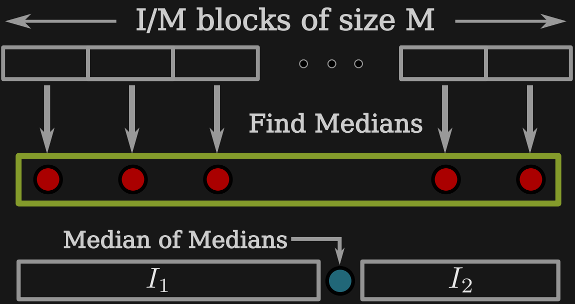
- Sequence of functions applied to input
  - No side effects
  - Easy to enforce *write-once* discipline
- Simple data-structures are difficult to implement
- Batch updates and node copying add to I/O and space complexity

- Graphs in the external model
  - Represented as **list of edges**
  - No adjacency list
- Semi-external Model
  - Vertices fit in internal memory
  - $|V| < M < |E|$

Building Blocks

- Selection
- Relabeling
- Contraction

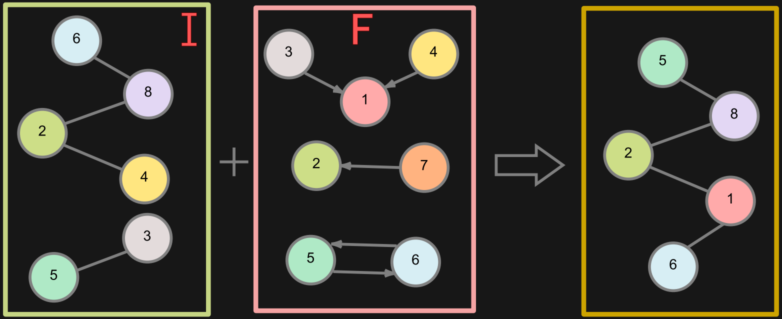
Select(I, k) using **median-of-medians** in $\mathcal{O}(scan(I))$



Recurse on either Select(I_1, k) or Select$(I_2, k - |I_1| - 1)$

In expectation, $T(N) = T\left(\frac{3}{4}N\right) + \mathcal{O}\left(\frac{N}{B}\right) \Rightarrow T(N) = \mathcal{O}\left(\frac{N}{B}\right)$

Relabel(I, F) in $\mathcal{O}(sort(I) + sort(F))$



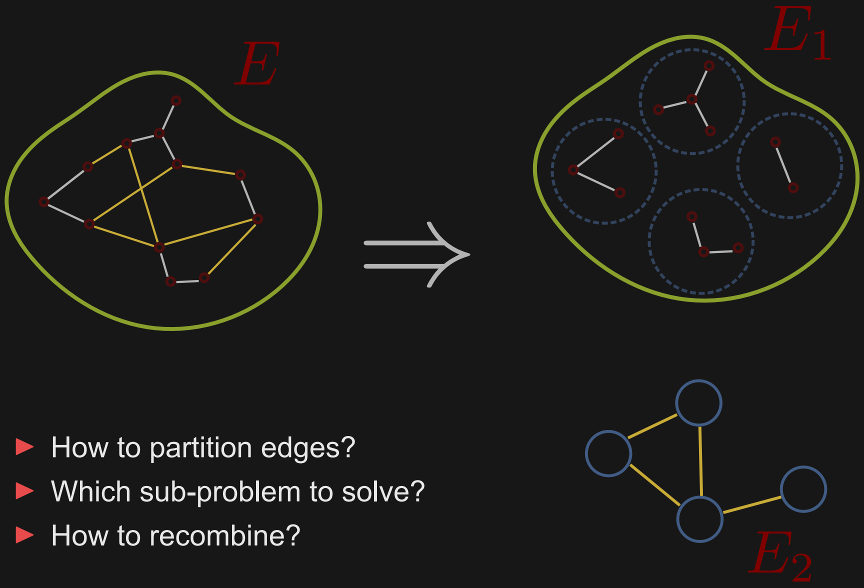
- Sort edges in I by first endpoint and edges in F by source
- Iterate through sorted lists in tandem, relabeling first endpoints of I
- Repeat for second endpoint by sorting I again

Contract$(I, \{C_1, C_2, \cdots\})$ in $\mathcal{O}\left(sort(I) + sort(\sum |C_i|)\right)$

Reduce to Relabeling

- Replace each C_i with a star S_i
- Concatenate $\langle S_1, S_2, \cdots \rangle$ to obtain F
- Contract$(I, \{C_1, C_2, \cdots\}$ = Relabel(I, F)

Graph Partitioning: Divide-and-Conquer



- How to partition edges?
- Which sub-problem to solve?
- How to recombine?

Connected Components

- Sample half the edges of the graph as E_1
- Recursively compute $\mathcal{C}_1 =$ Connected-Components$(G_1 = (V, E_1))$
- Use contraction to obtain $G_2 =$ Contract$(G, \mathcal{C}_1)$
- Recursively compute $\mathcal{C}_2 =$ Connected-Components(G_2)
- Return Connected-Components$(G) = \mathcal{C}_2 \cup$ Relabel $(\mathcal{C}_2, \mathcal{C}_1)$

$$T(N) = 2T\left(\frac{N}{2}\right) + \mathcal{O}(sort(|E|))$$

- Repeat until problem size $\leq M$
- $\log_2 \frac{|E|}{M}$ iterations
- Total I/O complexity $T(|E|) = \mathcal{O}\left(sort(|E|) \cdot \log_2 \frac{|E|}{M}\right)$

Minimum Spanning Tree

- Find the median edge weight m by running Select$(E, |E|/2)$
- Compute $E_1 \subset E$ as the set of edges with weight $\leq m$
- Recursively compute $T_1 =$ MST$(G_1 = (V, E_1))$
- Compute the connected components of the MST obtained using half the edges: $C_1 =$ Connected-Components(T_1)
- Use contracton to obtain $G_2 =$ Contract(G, C_1)
- Recursively compute $T_2 =$ MST(G_2)
- Return $T = T_1 \cup$ Inverse-Relabel(T_2, C_1)

Maximal Matching

- Sample half the edges of the graph as E_1
- Recurse to find $M_1 =$ Maximal-Matching$(G_1 = (V, E_1))$
- Find set of vertices covered by the matching $V_1 = V(M_1)$
- Let $E_2 = E \setminus (V_1 \times V_1)$ and $G_2 = (V, E_2)$
- Return $M = M_1 \cup$ Maximal-Matching(G_2)

Semi-external Model: Vertices fit in Internal Memory

Minimum Spanning Tree

- Maintain union-find data structure in memory
- Run Kruskal's algorithm

Connected Components

- How to re-arrrange edges contiguously by component?
- I/O complexity dominated by sorting: $\mathcal{O}\left(scan(|E|) \cdot \log_b \frac{|E|}{B}\right)$
- What if there are few connected components?
- Desired runtime: $\mathcal{O}\left(scan(|E|) \cdot \log_b |\mathcal{C}|\right)$
- $|\mathcal{C}|$ is # of connected components

Grouping N Elements with keys in range $[1 \cdots G]$

Use b blocks in internal memory

- Each block stores elements from a disjoint range of length G/b
- Blocks are emptied to external memory when full

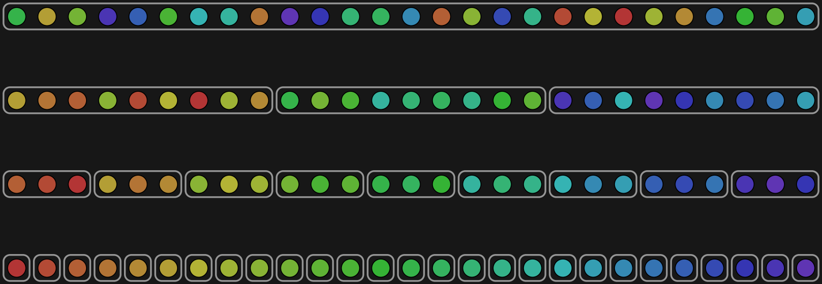
Recurse on each range (size G/b) from the last step

Sub-divide into three sub-ranges of size G/b^2 and so on . . .

Done after $\mathcal{O}(\log_b G)$ iterations

Total I/O complexity $= \mathcal{O}(scan(N) \cdot \log_b G)$

Grouping with $b = 3$ and $G = 27$



Partially filled blocks?

Concatenate to ensure that there is at most one.

Discussion

- Other graph problems:
  - Shortest paths
  - Random walk
- Assume properties of the ordering of edges

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A functional approach to external graph algorithms.
In *European Symposium on Algorithms*, pages 332–343. Springer, 1998.