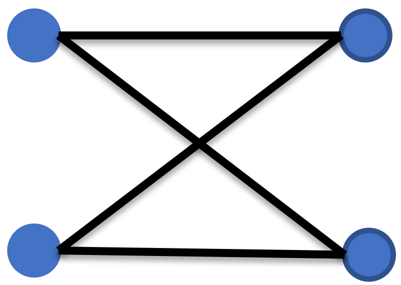
Parallel algorithms for butterfly computations

Jessica Shi (MIT CSAIL)
Julian Shun (MIT CSAIL)

What are butterflies?

Butterflies = 4-cycles = $K_{2,2}$



Think of these as the bipartite analogue of triangles (K_3)

Note: Bipartite graphs contain no triangles

Finding dense bipartite subgraphs

- Finding dense subgraphs: (not bipartite)
  - K-core: Repeatedly find + delete min degree vertex
  - Triangle peeling (triangle densest subgraph): approx by repeatedly find + delete vertex containing min # triangles

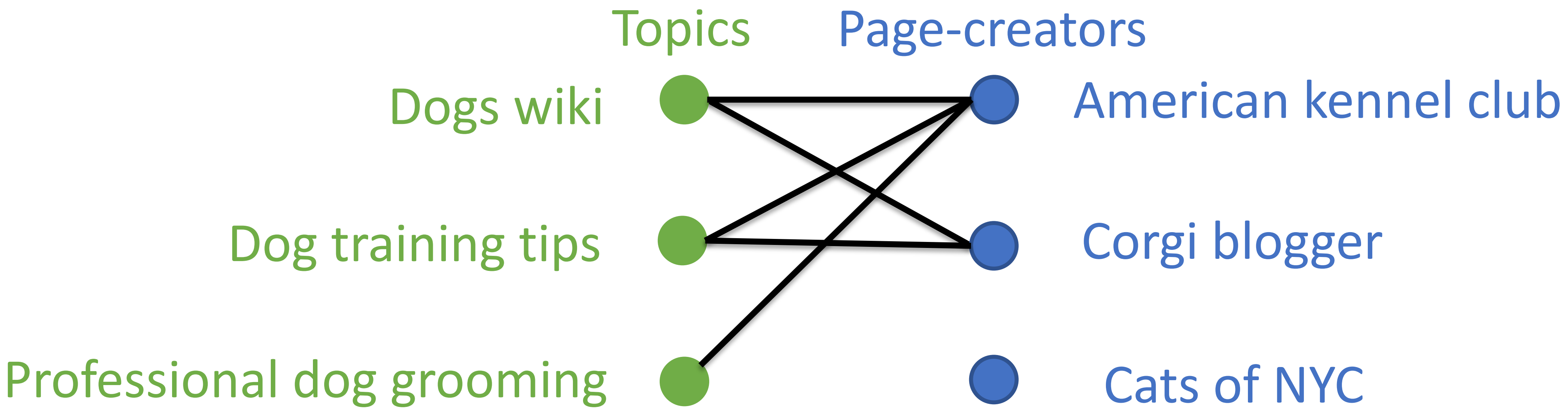
- Butterfly peeling: Repeatedly find + delete vertex containing min # of butterflies[1]

- Applications:
  - Link spam detection: External links to a spam page, for self-promotion in search rankings

[1] Sariyuce and Pinar (18)

Link spam detection

- Nodes = webpages, Edges = links
- Web communities = dense bipartite subgraphs
  - Bipartitions = topics, page-creators interested in topics



Outline

- Main goal: Build a framework ParButterfly to count and peel butterflies

- New parallel algorithms for butterfly counting + peeling
- ParButterfly framework with modular settings
  - Tradeoff b/w theoretical bounds + practical speedups
- Comprehensive evaluation
  - Counting outperforms fastest seq algorithms by up to 13.6x
  - Peeling outperforms fastest seq algorithms by up to 10.7x

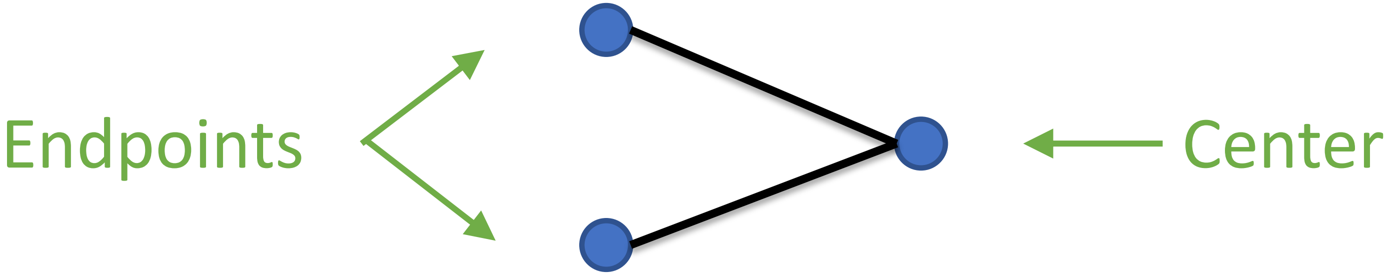
Important paradigms

- Parallelization
  - Shared memory
  - Work-span model:
    - Work = total # operations
    - Span = longest dependency path
- Strong theoretical bounds
  - Work-efficient = work matches sequential time complexity
- Fast in practice

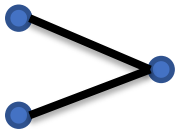
ParButterfly counting framework

How do we count butterflies? (per vertex)

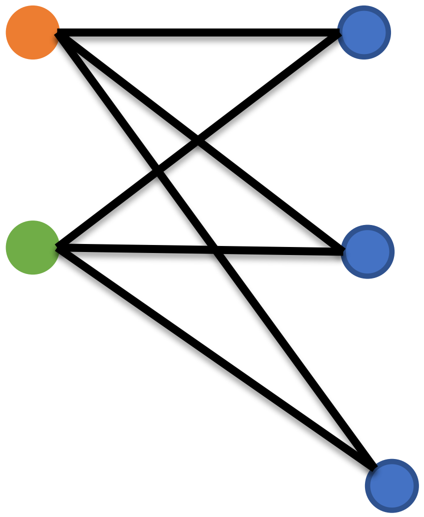
Wedge = P_2 =



How do we count butterflies? (per vertex)

Wedge = P_2 = 

Wedges with the same endpoints form butterflies:



wedges w/endpoints = w = 3

butterflies on endpoints = $\binom{w}{2} = \binom{3}{2}$ = 3

butterflies on each center = $w - 1$ = 3 – 1 = 2

Counting framework so far

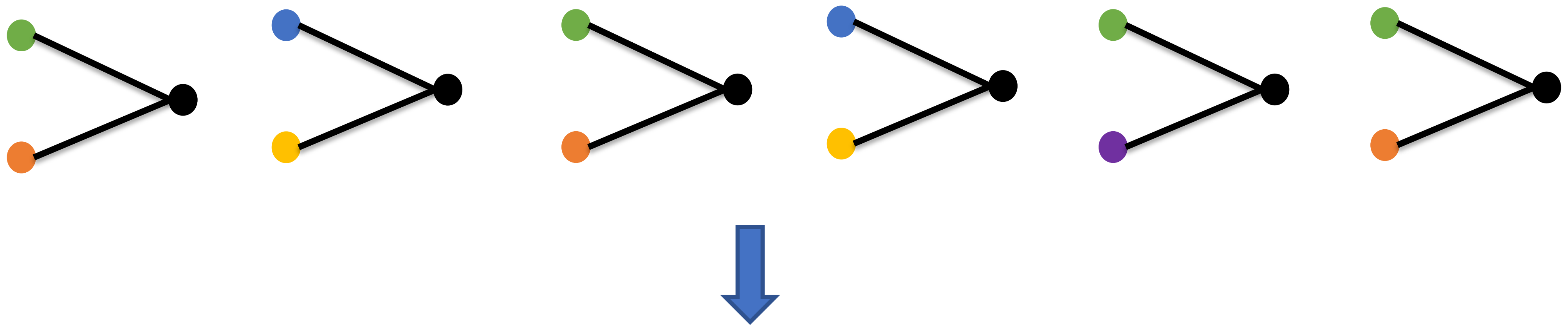
1. Retrieve wedges
2. Aggregate wedges: For each pair of endpoints, count # wedges w
3. Compute butterfly counts:
   - $+\binom{w}{2}$ for each endpoint
   - $+\, w - 1$ for each center

One question: How do we aggregate wedges?

(will discuss wedge retrieval after)

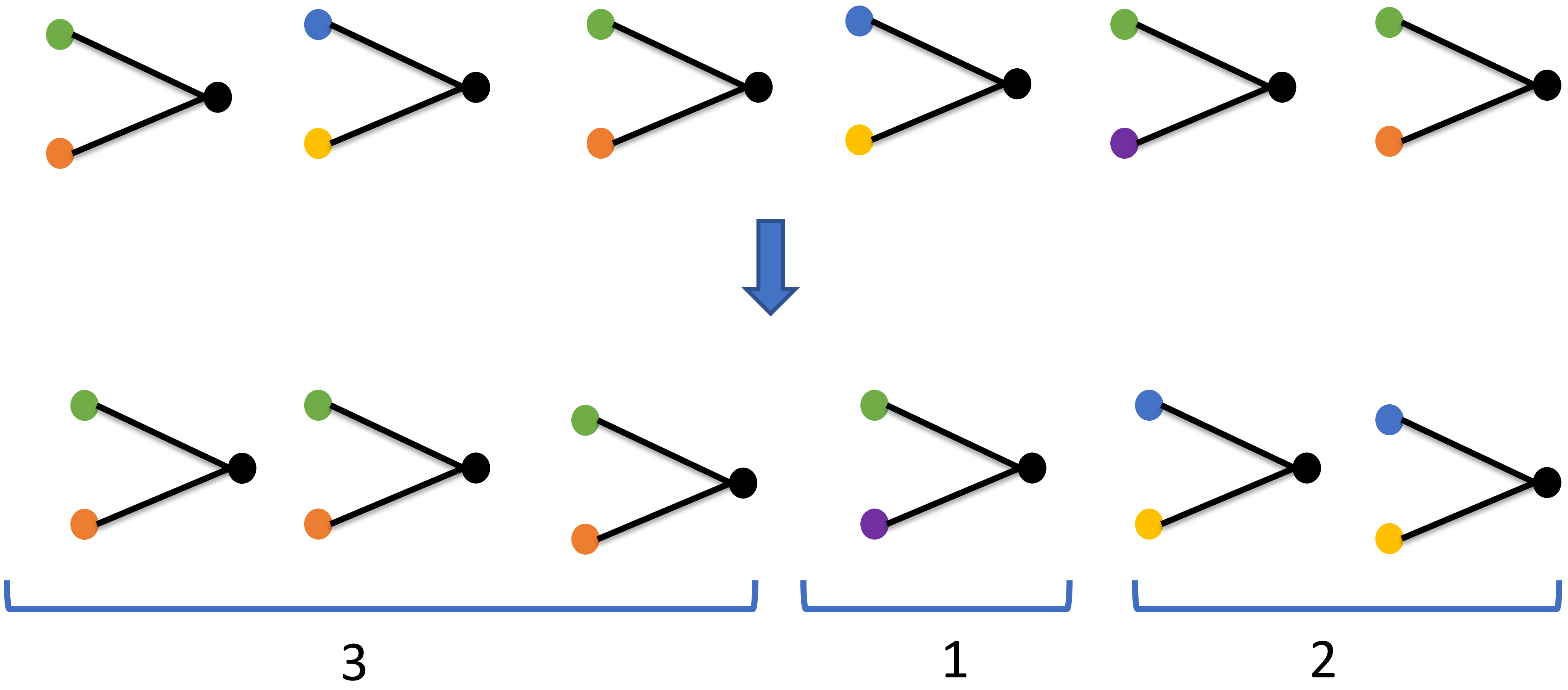
Wedge aggregating

- Method 1: Semisorting (on endpoints)



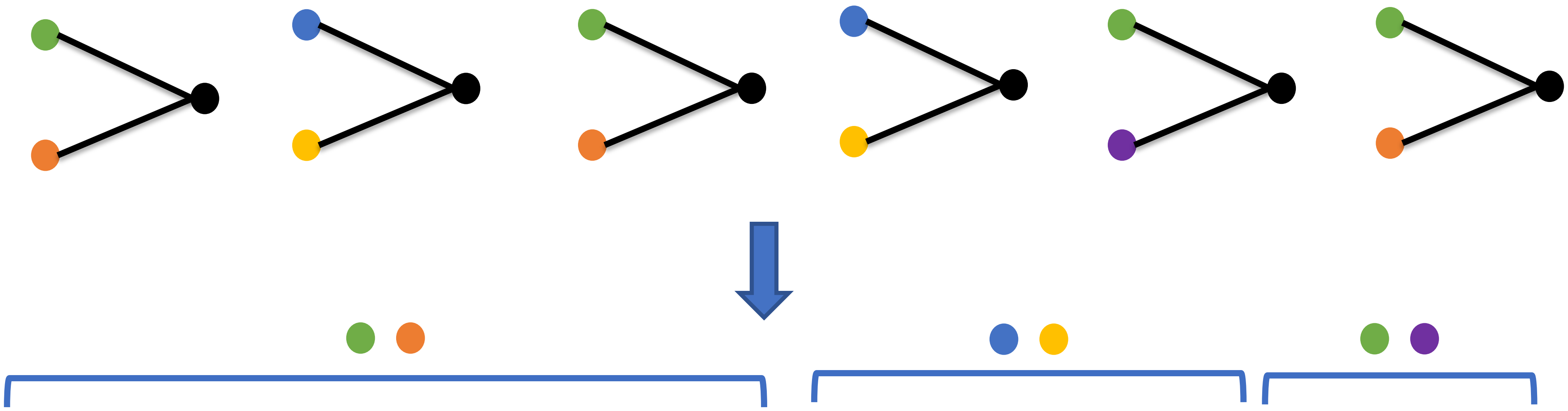
Wedge aggregating

- Method 1: Semisorting (on endpoints)



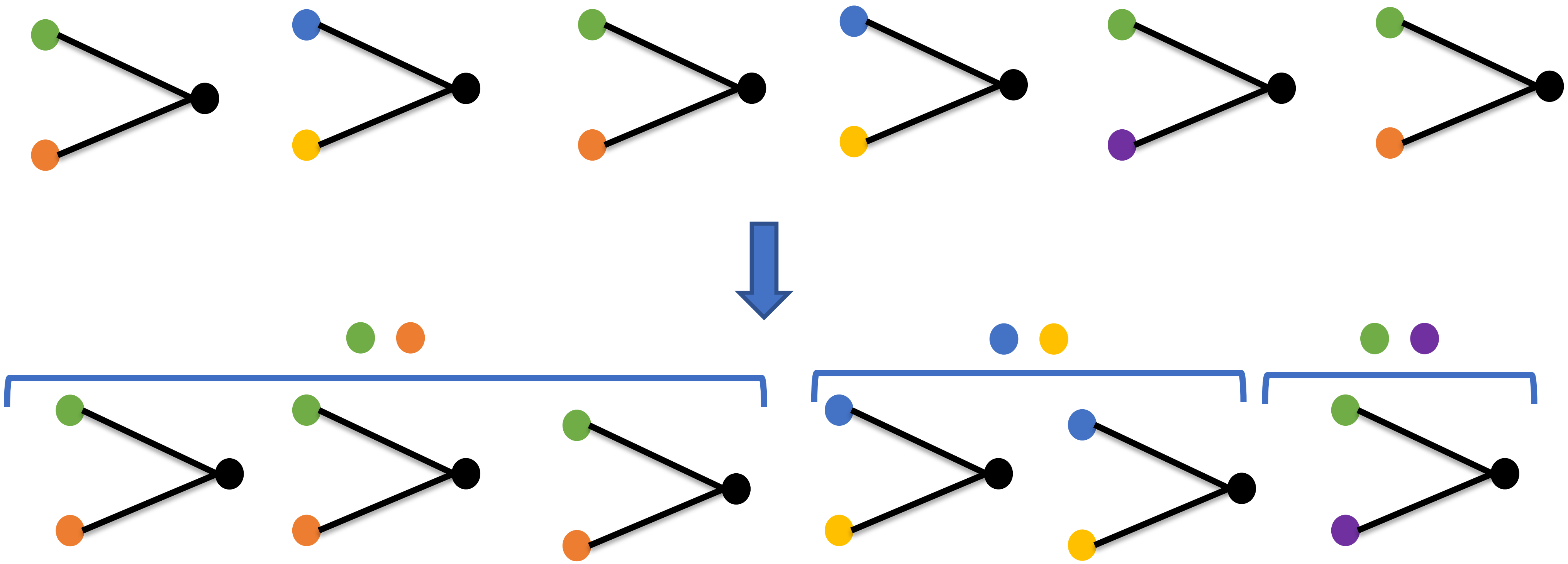
Wedge aggregating

- Method 2: Hashing (keys = endpoints)



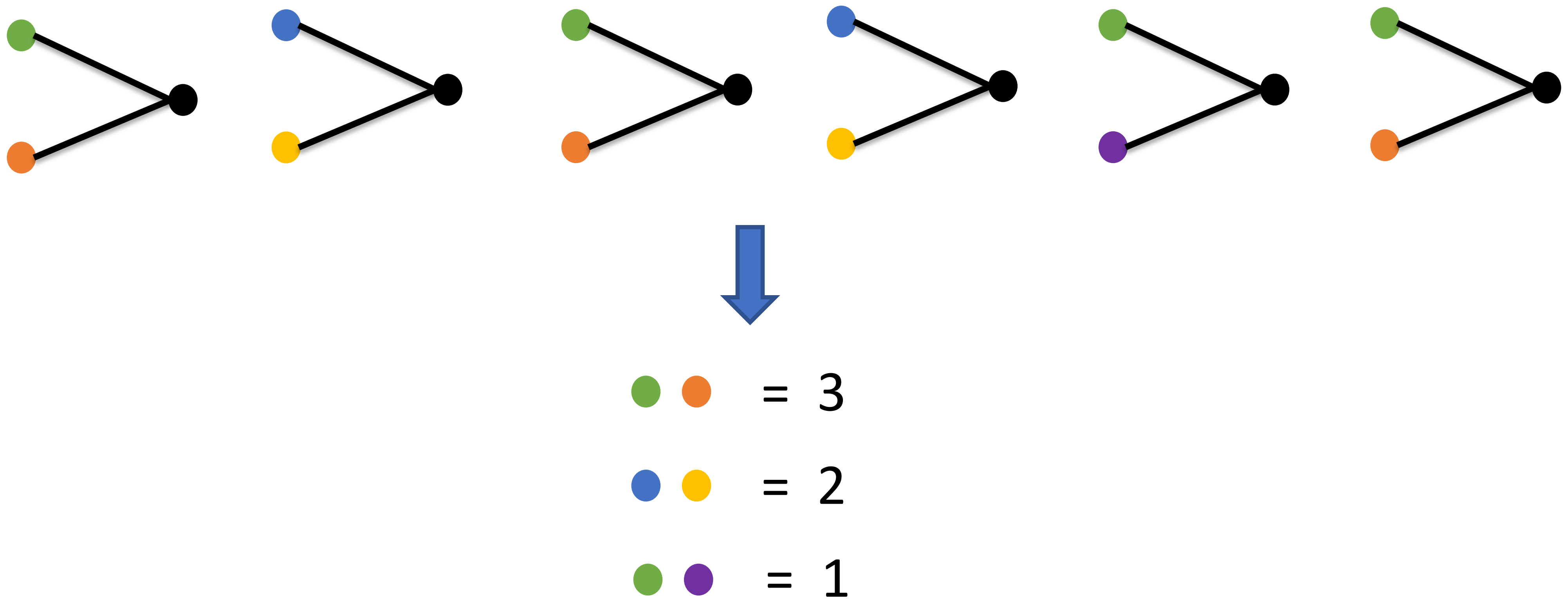
Wedge aggregating

- Method 2: Hashing (keys = endpoints)



Wedge aggregating

- Method 3: Histogramming (frequencies of endpoints)



Wedge aggregating bounds

Semisorting[1], hashing[2], and histogramming[3] are all work-efficient

w = # of wedges

O(w) expected work, O(log w) span whp

[1] Gu, Shun, Sun, and Blelloch (15)
[2] Shun and Blelloch (14)
[3] Dhulipala, Blelloch, and Shun (17)

Butterfly counts from wedge counts

Each wedge produces butterfly counts per vertex

Another question: How do we handle butterfly counts on the same vertex in parallel?

1. Use atomic adds
2. Aggregate counts in the same way we aggregated wedge counts (semisorting, hashing, histogramming)

Counting framework so far

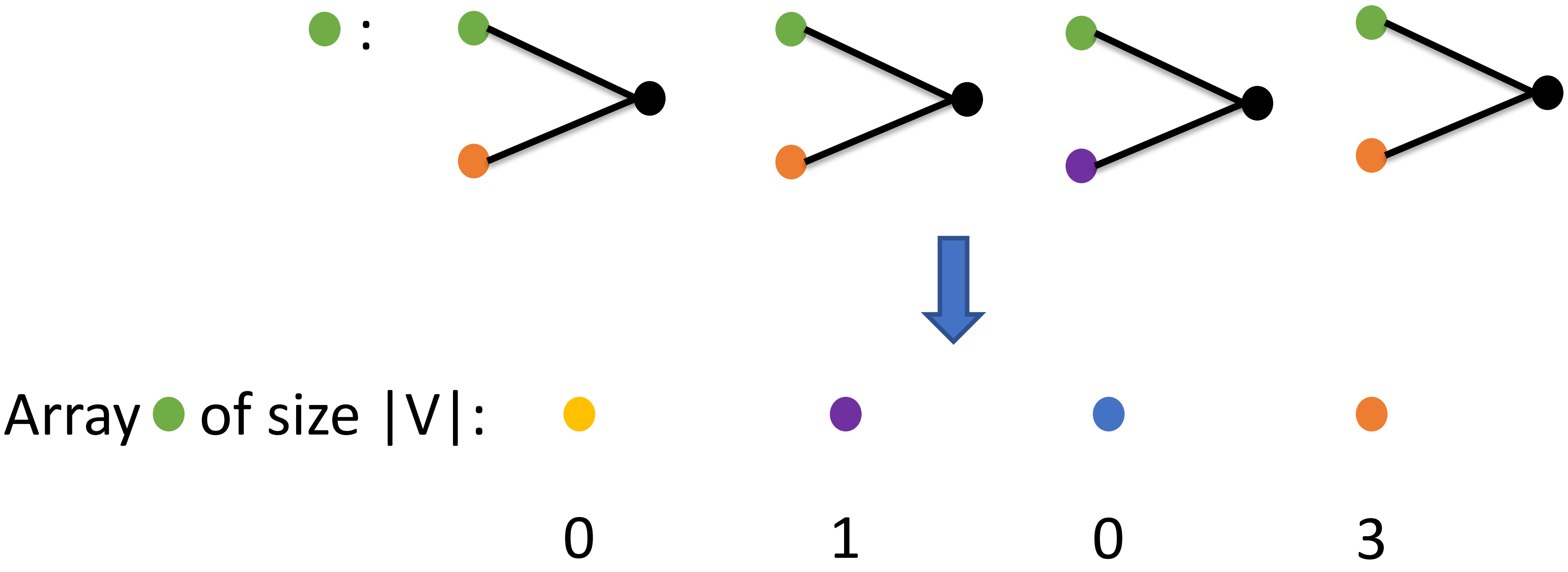
1. Retrieve wedges
2. Aggregate wedges:
   - Semisort, Hash, Histogram
3. Compute butterfly counts:
   - Semisort, Hash, Histogram, Atomic add

One more way to count wedges: Batching

(not with polylogarithmic span, but fast in practice)

Wedge aggregating (batching)

- Main idea: Process a subset of vertices in parallel, finding all wedges where those vertices are endpoints



Wedge aggregating (batching)

- Each vertex requires linear memory →
- How many vertices do we process in parallel?
  - Simple: Fixed # based on memory available
  - Wedge-aware: Dynamically choose based on how many wedges will be processed per vertex

Counting framework so far

1. Retrieve wedges
2. Aggregate wedges:
   - Semisort, Hash, Histogram, Batch (Simple + Wedge-aware)
3. Compute butterfly counts:
   - Semisort, Hash, Histogram, Atomic add

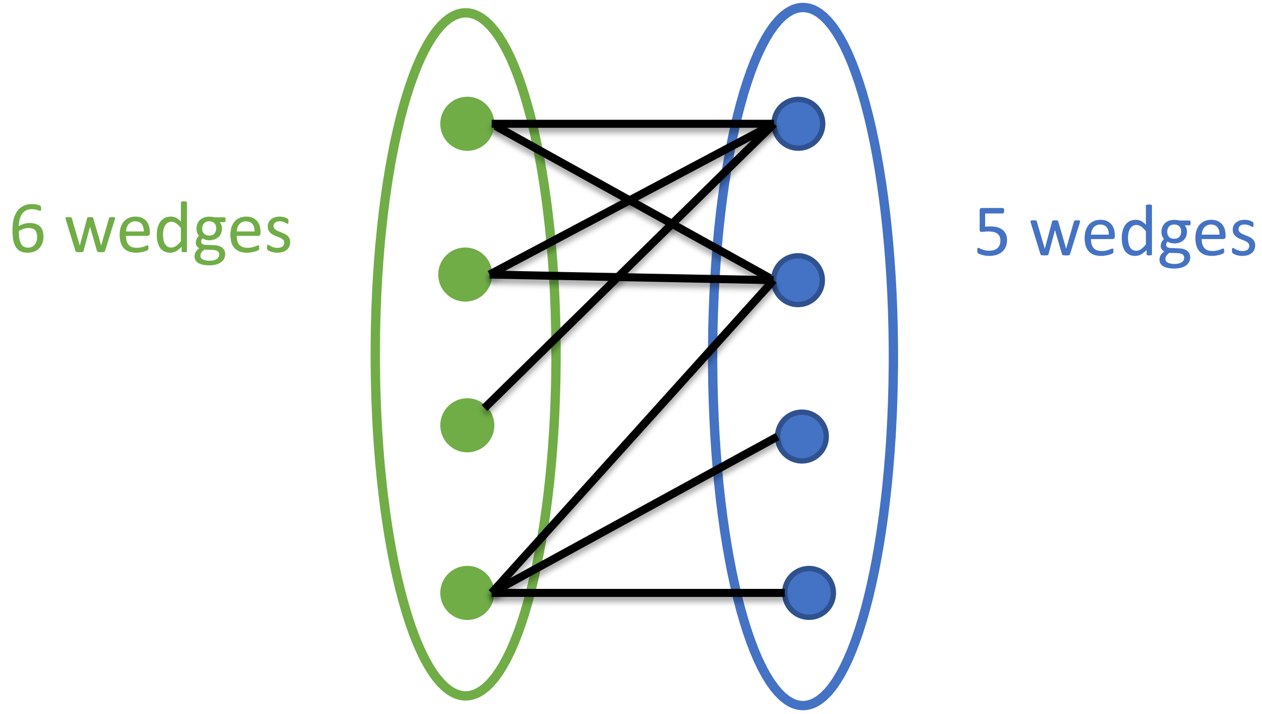
More questions:

How do we retrieve wedges?

How many wedges are there?

It depends!

- Method 1: Process wedges w/endpoints from one bipartition (Side) [1]



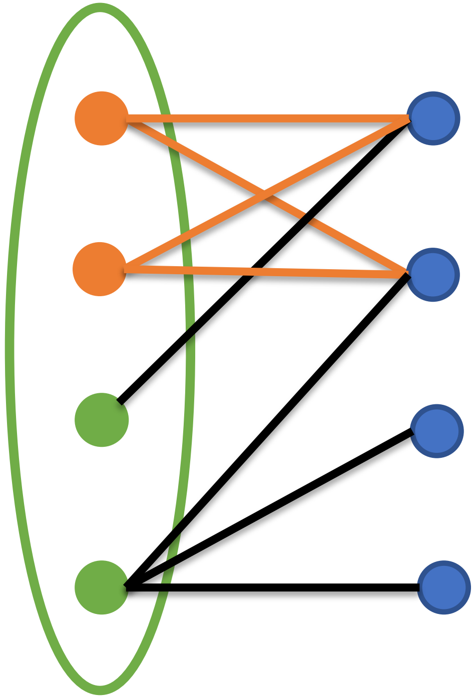
Is this optimal (min # wedges)? Not always.

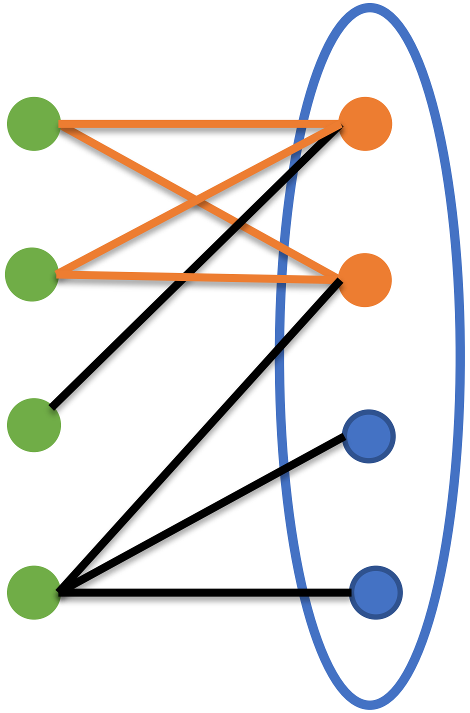
[1] Sanei-Mehri, Sariyuce, Tirthapura (18)

(Note: Butterfly count remains the same)

- Regardless of which side we pick, butterfly count does not change – only some “useful” wedges create butterflies

6 wedges





5 wedges

2 “useful” wedges = 1 butterfly

2 “useful” wedges = 1 butterfly

Retrieve wedges

- Method 2: Degree ranking

Main idea:

Once we obtain all wedges with endpoint v, we do not have to consider wedges with endpoint v again.

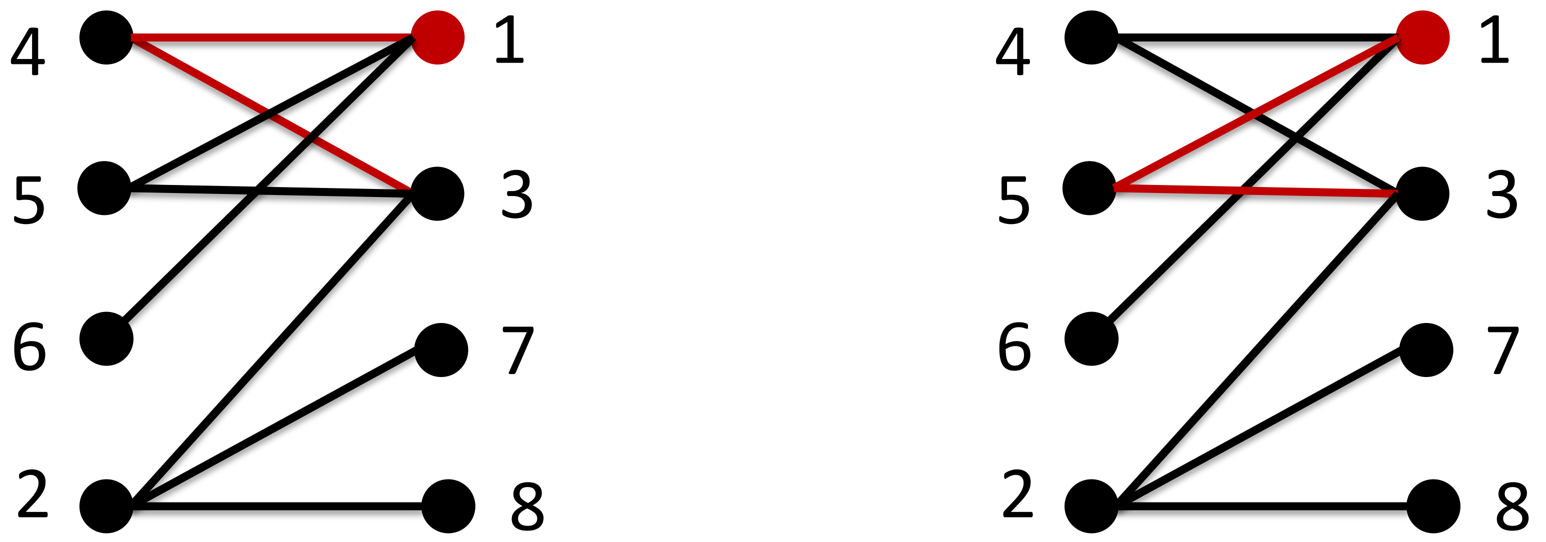
Retrieve wedges

- Method 2: Degree ranking

1. Order vertices by non-increasing degree
2. For each vertex v, only consider wedges with endpoint v that is formed by vertices later in the ordering than v

Retrieve wedges

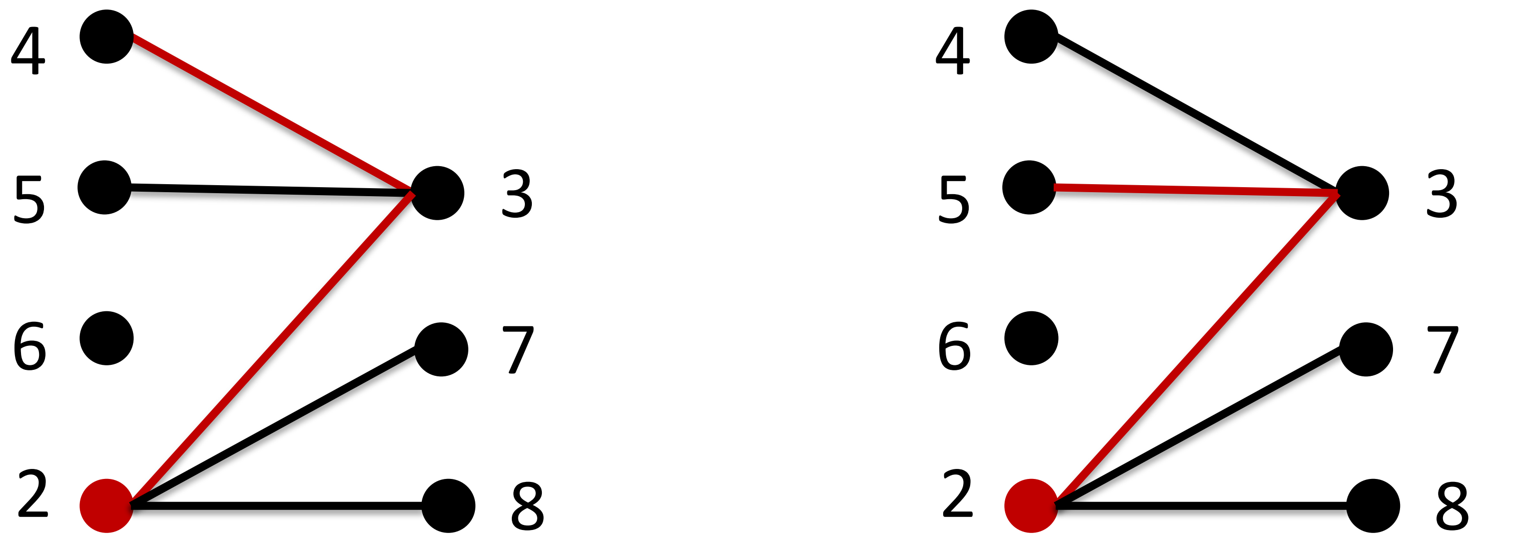
- Method 2: Degree ranking



2 wedges

Retrieve wedges

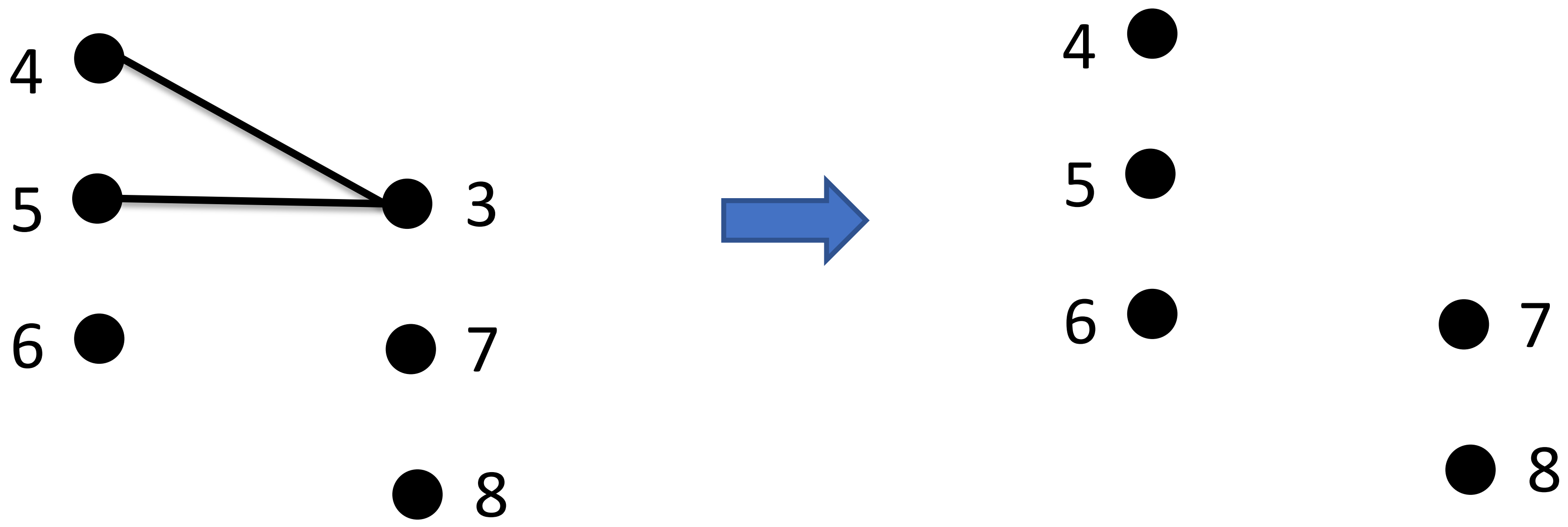
- Method 2: Degree ranking



2 wedges

Retrieve wedges

- Method 2: Degree ranking



We only processed 4 wedges!

Degree ranking [1]

- # wedges processed using degree order = O(αm)
  - α = arboricity (O($\sqrt{m}$))
  - m = # edges
- Therefore: (using work-efficient options)

Ranking vertices = O(m) expected work, O(log m) span whp
Retrieving wedges = O(αm) expected work, O(log m) span whp
Counting wedges = O(αm) expected work, O(log m) span whp
Computing butterfly counts = O(αm) expected work, O(log m) span whp

Total = O(αm) expected work, O(log m) span whp

[1] Chiba and Nishizeki (85)

Other rankings

- Approximate degree order
  - Log degree
- Complement degeneracy order
  - Ordering given by repeatedly finding + deleting greatest degree vertex
- Approximate complement degeneracy order
  - Complement degeneracy order, but using log degree

We show these are all work-efficient

Counting framework

1. Rank vertices:
   - Side, Degree, Approx Degree, Co Degeneracy, Approx Co Degeneracy
2. Retrieve wedges
3. Aggregate wedges:
   - Semisort, Hash, Histogram, Batch (Simple + Wedge-aware)
4. Compute butterfly counts:
   - Semisort, Hash, Histogram, Atomic add

$O(\alpha m)$ expected work, $O(\log m)$ span whp

ParButterfly peeling framework

How do we peel butterflies?

- Goal: Iteratively remove all vertices with min butterfly count

Subgoal 1: A way to keep track of vertices with min butterfly count
Subgoal 2: A way to update butterfly counts after peeling vertices

Note: We've already done subgoal 2 in counting framework

For subgoal 1, we give a work-efficient batch-parallel Fibonacci heap which supports batch insertions/decrease-keys (see paper).

Peeling framework

1. Obtain butterfly counts
2. Iteratively remove vertices with min butterfly count
   - Use batch-parallel Fibonacci heap to find vertex set S
   - Count wedges with endpoints in S
     - Semisort, Hash, Histogram, Batch (Simple + Wedge-aware)
   - Compute updated butterfly counts
     - Semisort, Hash, Histogram

Peeling framework bounds

- By vertex: (ρ_v = number of peeling rounds across all vertices)

$O(\rho_v \log m + \sum \text{degree}(v)^2)$ expected work, $O(\rho_v \log^2 m)$ span whp

- By edge: (ρ_e = number of peeling rounds across all edges)

$O(\rho_e \log m + \sum_{(u,v)} \sum_{u' \in N(u)} \min(\text{degree}(u), \text{degree}(u')))$ expected work,

$O(\rho_e \log^2 m)$ span whp

Evaluation

Environment

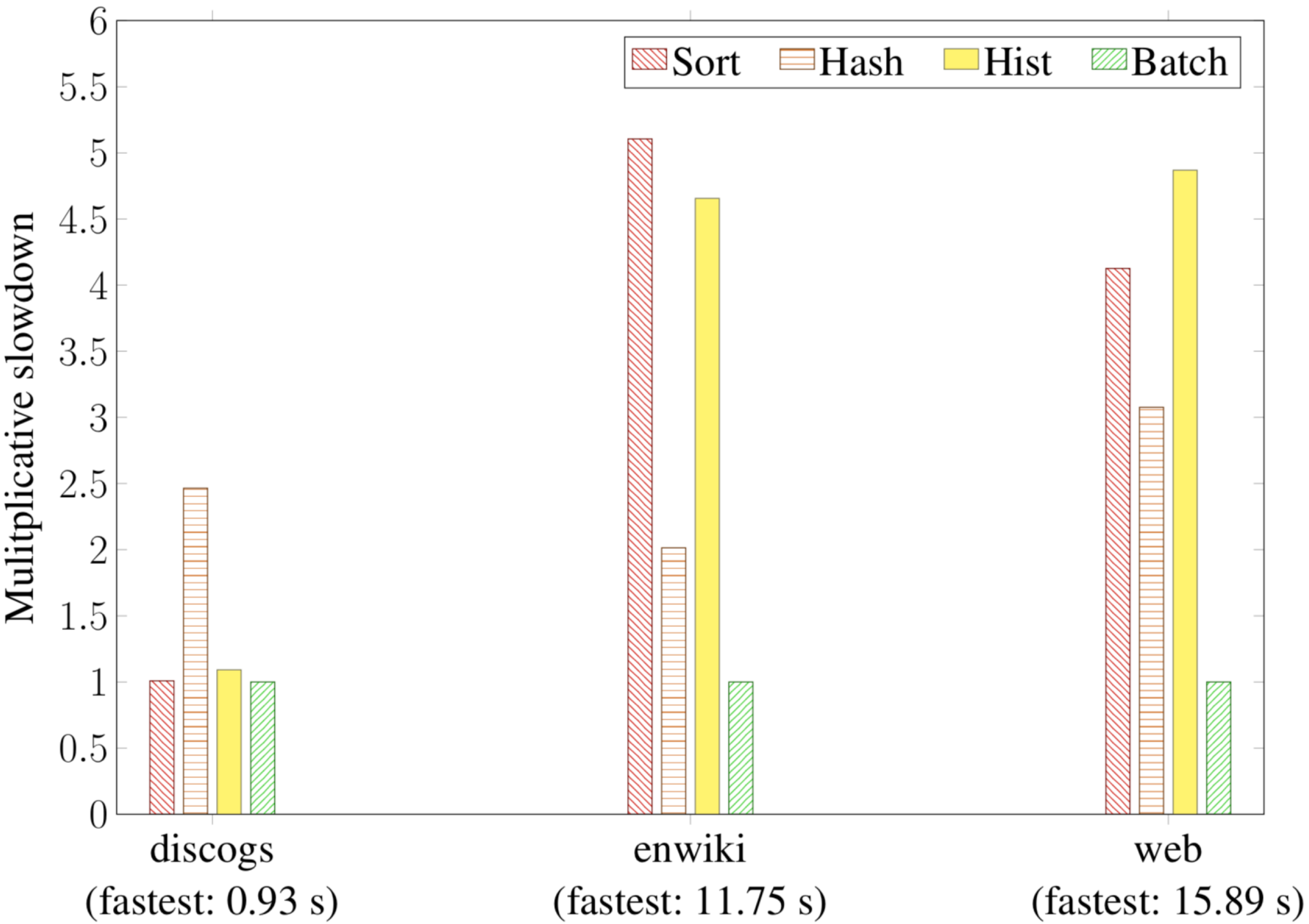
- m5d.24xlarge AWS EC2 instance: 48 cores (2-way hyper-threading), 384 GiB main memory
- Cilk Plus[1] work-stealing scheduler
- Koblenz Network Collection (KONECT) bipartite graphs

- Some modifications:
  - Julienne[2] instead of batch-parallel Fibonacci heap
  - Cannot hold all wedges in memory – batch wedge retrieval

[1] Leiserson (10)
[2] Dhulipala, Blelloch, and Shun (17)

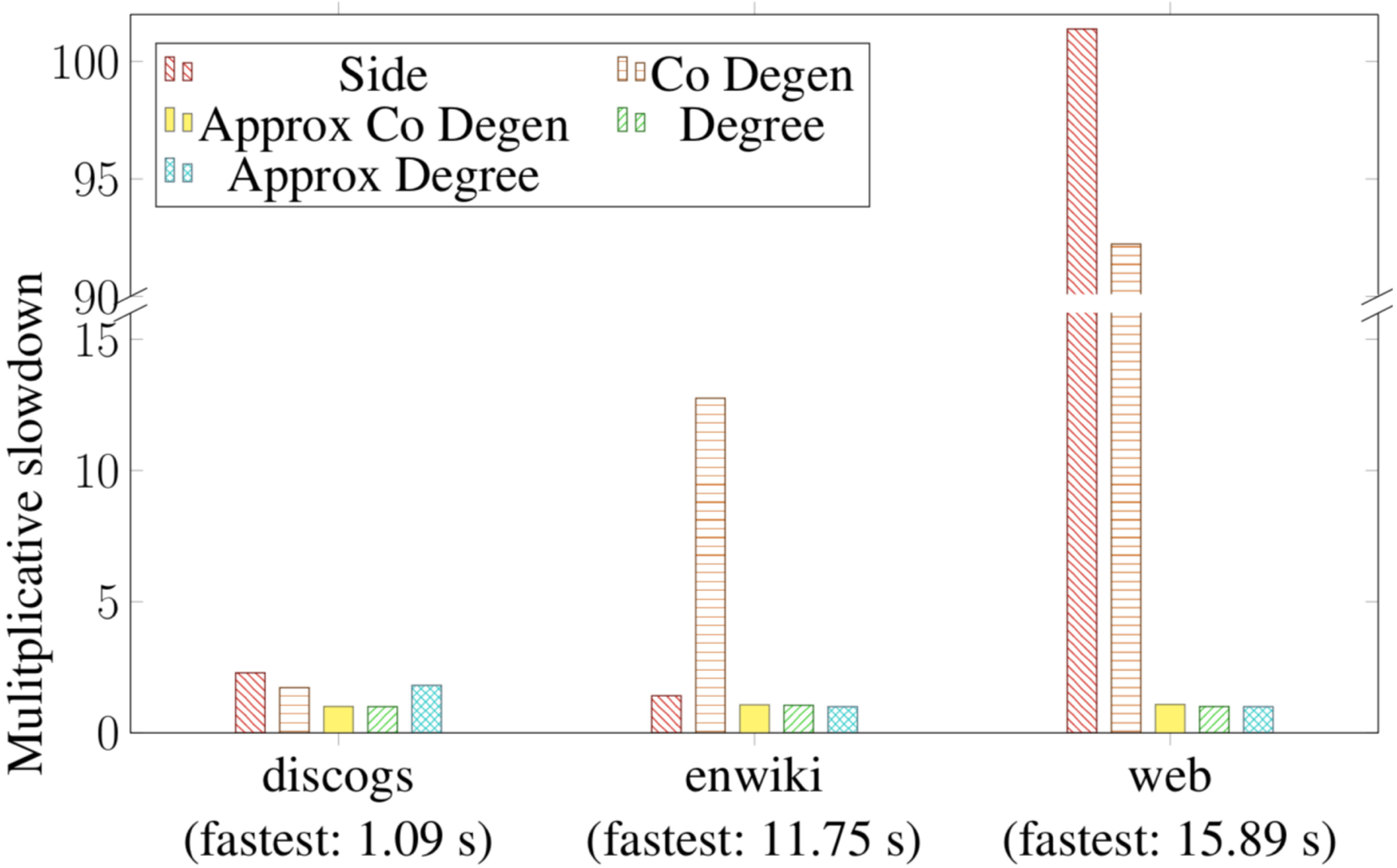
Counting: Best aggregation method:

Batching



Counting: Best ranking method:

Approx Complement Degeneracy / Approx Degree



Butterfly counting results

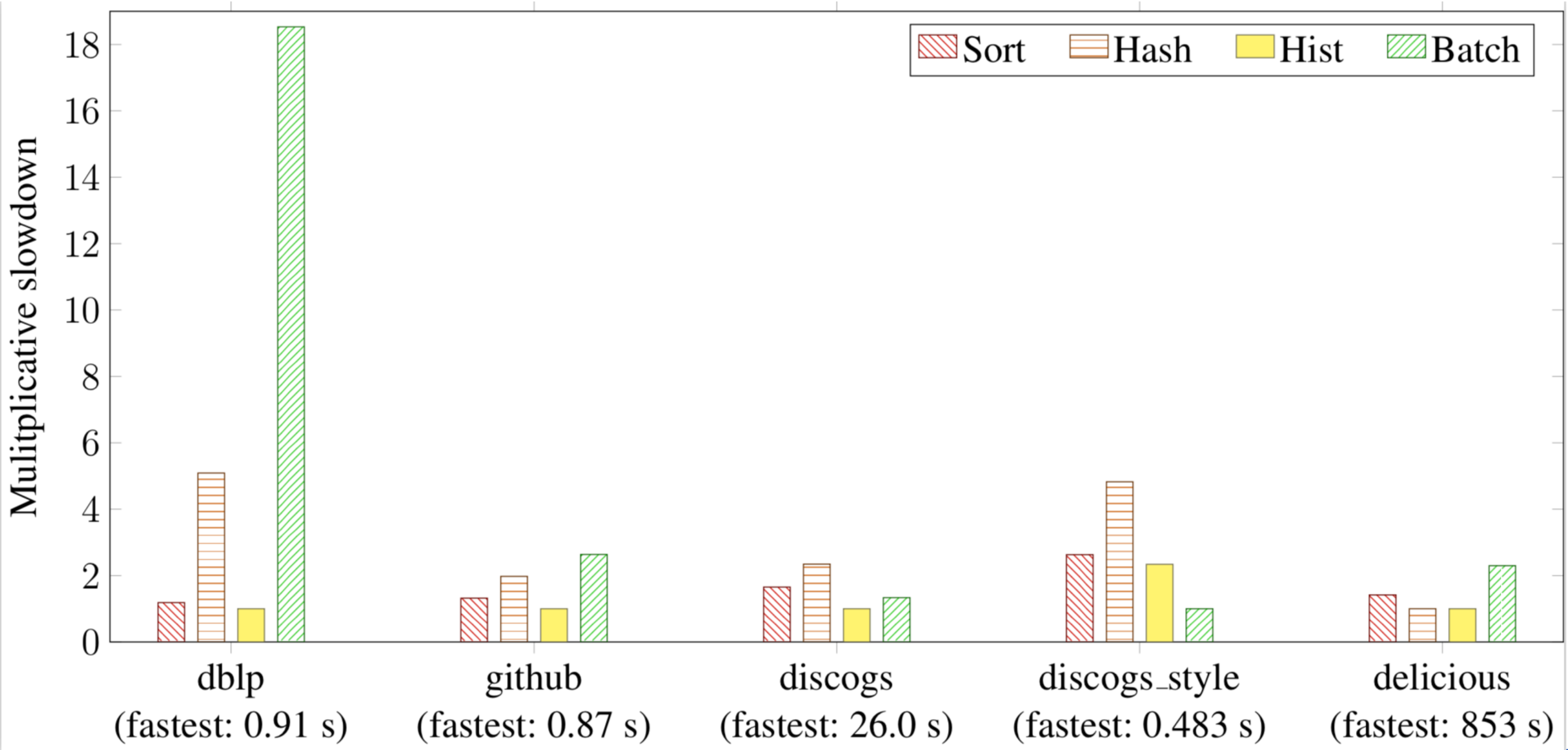
- 6.3 – 13.6x speedups over best seq implementations[1] [2]
- 349.6 – 5169x speedups over best parallel implementations[3]
  - Due to work-efficiency
- 7.1 – 38.5x self-relative speedups

- Up to 1.7x additional speedup using a cache-optimization[4]

[1] Sanei-Mehri, Sariyuce, Tirthapura (18)
[2] ESCAPE: Pinar, Seshadhri, Vishal (17)
[3] PGD: Ahmed, Neville, Rossi, Duffield, and Wilke (17)
[4] Wang, Lin, Qin, Zhang, and Zhang (19)

Peeling: Best aggregation method:

Histogramming



Butterfly peeling results

- 1.3 – 30696x speedups over best seq implementations[1]
  - Depends heavily on peeling complexity
  - Largest speedup due to better work-efficiency for some graphs
- Up to 10.7x self-relative speedups
  - No self-relative speedups if small # of vertices peeled

[1] Sariyuce and Pinar (18)

Conclusion

Conclusion

- New parallel algorithms for butterfly counting/peeling
- Modular ParButterfly framework w/ranking + aggregation options
- Strong theoretical bounds + high parallel scalability
- Github: https://github.com/jeshi96/parbutterfly

- Future work:
  - Cycle counting extensions
  - Better work bounds for butterfly peeling

Thank you

Priority queue for butterfly counts

Batch-parallel Fibonacci heap:

- *k* insertions: O(*k*) amortized expected work, O(log(*n*+*k*)) span whp
- *k* decrease-keys: O(*k*) amortized work, O($\log^2 n$) span whp
- delete-min: O(log *n*) amortized expected work, O(log *n*) span whp