Engineering a Cache-Oblivious Sorting Algorithm

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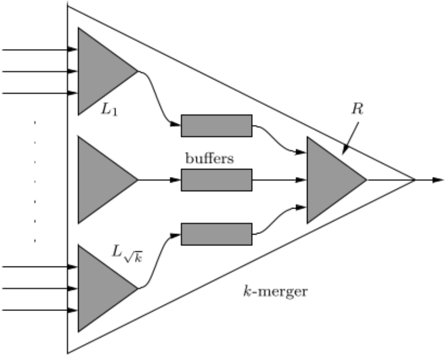
Overview

Food for thought...

1. In what ways did the results of one experiment critically determine the parameters for a later one?
2. What hypotheses did the authors have? Which of these seem sensible but are not supported by the experiments?
3. How do the authors ensure that their experiments are robust, reliable, and reproducible? What do you find unusual?
4. How could some of these results have been discovered with the help of tuning tools like OpenTuner?

k-merger

- Central to (lazy) funnelsort
- Recursively built of $\sqrt{k}$-mergers
- Outputs of mergers on one level are inputs to parents
- When buffers are empty, recursively invoke the filling algorithm

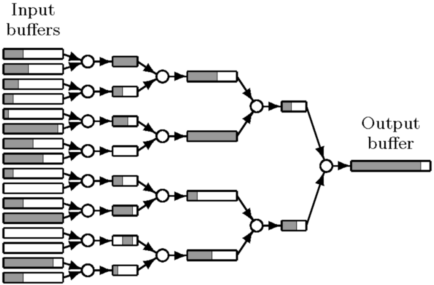


Funnelsort on n elements incurs $O(1 + \frac{n}{B}(1 + \log_M n))$

Modified k-merger

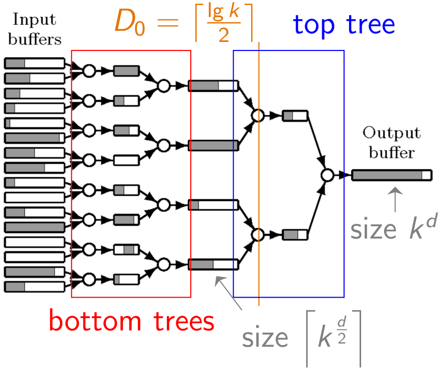
From prior work by Brodal and Fagerberg [1]:

```
Fill(v)
while v's output buffer isn't full
    if left input buffer empty
        Fill(left child of v)
    if right input buffer empty
        Fill(right child of v)
    perform one merge step
```



Modified k-merger

Lemma 1. *Let $d \geq 2$. The size of a k-merger (excluding its output buffer) is bounded by $c \cdot k^{\frac{d+1}{2}}$ for a constant $c \geq 1$. Assuming $B^{\frac{d+1}{d-1}} \leq \frac{M}{2c}$, a k-merger performs $O\left(\frac{k^d}{B}\log_M(k^d) + k\right)$ I/Os during an invocation.*

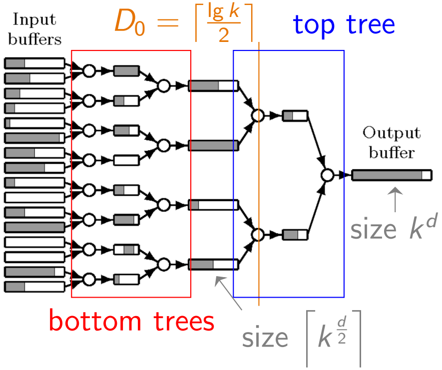


Modified k-merger

Space bound:

$$S(k) = k^{\frac{1}{2}} \cdot k^{\frac{d}{2}} + (k^{\frac{1}{2}} + 1) \cdot S(k^{\frac{1}{2}})$$
$$\leq c \cdot k^{\frac{d+1}{2}}$$

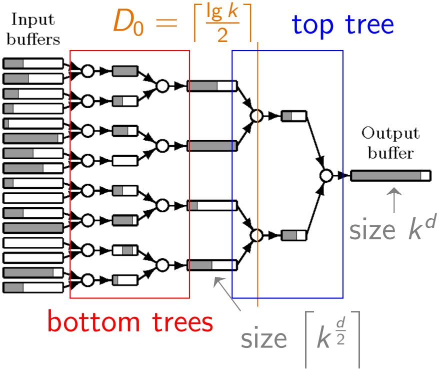
- $k^{\frac{1}{2}}$ buffers with $k^{\frac{d}{2}}$ size
- Recurse on $k^{\frac{1}{2}}$-size problems
  - 1 recursion "up"
  - $k^{\frac{1}{2}}$ recursions "down"



Modified k-merger

I/O bound:

1. Largest subtree with $\bar{k}$ leaves
   - Space: $\bar{k}^{\frac{d+1}{2}} \leq \frac{M}{2c}$
2. Parent has $\bar{k}^2$ leaves
   - Space: $(\bar{k}^2)^{\frac{d+1}{2}} > \frac{M}{2c}$
   - Input: "large buffers"
3. Remove large buffer edges
   - Connected *base trees*



Modified k-merger

4. 1 block $\times$ $\bar{k}$ buffers: $\bar{k}B \leq \left(\frac{M}{2c}\right)^{\frac{2}{d+1}} \cdot \left(\frac{M}{2c}\right)^{\frac{d-1}{d+1}} \leq \frac{M}{2c}$
5. Base tree + 1 block $\times$ $\bar{k}$ buffers $\leq \frac{M}{c}$ space
6. If k-merger is a base tree, output k^d items in $O\left(\frac{k^d}{B} + k\right)$ I/Os
7. Otherwise, for Fill(v = root node of a base tree)
   - a. Loads $\Omega(\bar{k}^d)$ elements to output buffer
   - b. Base tree + 1 block/buffer $= O\left(\frac{1}{B}\bar{k}^{\frac{d+1}{2}} + \bar{k}\right)$
   - c. $\bar{k}^{d+1} > \frac{M}{2c} \implies \bar{k}^{d-1} > \left(\frac{M}{2c}\right)^{\frac{d-1}{d+1}} \geq B \therefore \frac{1}{B}\bar{k}^d \geq \bar{k}$
   - d. Casework: recursive calls may cause base tree reloads
   - e. Charge $O\left(\frac{1}{B}\right)$ I/O per large buffer insert
   - f. $\geq \left(\frac{M}{2c}\right)^{\frac{1}{d+1}}$ leaves means $O(\log_M k^d)$ large buffer inserts/item

Modified k-merger

- Per invocation:
  - k-merger is base tree: $O\left(\frac{k^d}{B} + k\right)$ I/Os
  - k-merge is not: $O(\frac{k^d}{B} \log_M k^d)$ large buffer I/Os
  - Overall I/O cost bounded by $O(\frac{k^d}{B} \log_M k^d + k)$
- Proof based on buffer size, any memory layout works!
- This paper: $D_0/D_0{+}1$ buffers have size $\alpha\lceil d^{\frac{3}{2}}\rceil$, $\alpha > 0$

Ingredients

Cache-oblivious sorting implementation

- 1 baseline cache-oblivious sorting algorithm, theoretically efficient
- 1 state-of-the-art sorting algorithm, not necessarily cache-oblivious
- 1 or more workloads, aiming to cover useful sorting applications
- 1 or more data distributions, to simulate different workload types
- 1 consistent method for accurately measuring time
- Several machines and architectures, optional but recommended
- Many hypotheses that can translate into experiments

Step 1: k-merger structures

Allocator

- Custom
- Standard*

Invocation pattern

- Recursive
- Iterative

Navigation

- Pointers
- Implicit

Layout

- BFS
- DFS
- vEB

Merger nodes

- stored with output buffer
- stored separately

Step 1: k-merger structures

Experiments

- Cartesian product of factors
- 28 experiments on 3 machines
- Workload
  - k streams of k^2 items
  - k-merger: $(\alpha, d)=(1, 2)$
  - Basic merger degree(z): 2
  - $k \in [15, 270]$
  - Measure $\lceil \frac{20000000}{k^3} \rceil$ merges

Step 1: k-merger structures

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- ✓ Standard*

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Step 2: Tuning basic mergers

Idea: improve on the merge step of the basic mergers

- Basic merger
- Coarse bound-checking
- Hybrid bound-checking
- Hwang-Lin merging algorithm [2]
- Hybrid Hwang-Lin

Experiments:

- Same as step 1, but with three additional (α, d) pairs:
  - $(1, 3)$
  - $(4, 2.5)$
  - $(16, 1.5)$

Step 2: Tuning basic mergers

Results:

- Hwang-Lin has a large overhead
- Bound-checking is ineffective
- Hybrids work better
- ✓ Straightforward works best

CPU branch prediction is really good, hand-coding is just extra overhead

Step 3: Degree of basic mergers

Idea: multiway mergers: less data movement, more complex

- Basic mergers
- Various multiway mergers
- Looser trees [3]

Experiments:

- $(k, \alpha, d) = (120, 16, 2)$
- 8 mergings of 1728000 elements
- $z \in [2, 9]$

Step 3: Degree of basic mergers

Results:

- 4- and 5-way mergers work best
- Looser trees don't show inflection, but have high overhead

Step 4: Caching for basic mergers

Idea: construct one k-merger per level

- Each level uses the same size k-merger
- Reset and reuse the k-merger for merging in the same level

Experiments:

- $(\alpha, d, z) = (4, 2.5, 2)$
- Straightforward binary basic mergers
- Base case uses std::sort() for sizes $< \alpha z^d = 23$
- Workloads: between $[5000000, 200000000]$ elements

Results: 3-5% speedup across the board

Step 5: Base sorting algorithm

Idea: choose a good base case for sorting a small number of elements
Experiments:

- Insertion, selection, heap, shell, and std::sort() sorts
- Workload: input sizes from 10 to 100

Results: std::sort() is fastest

Step 6: Parameters α and d

Idea: choose good α and d parameters

Experiments:

- $\alpha \in [1, 40]$
- $d \in [1.5, 3]$
- Workloads: various sizes

Results:

- $\alpha < 10$ produces a longer running time
- d does not have a large impact at reasonable sizes
- Small (α, d) correspond to small buffer sizes
  - Cost of navigation and invocation spread over fewer merge steps
- Optimal $(\alpha, d) \approx (16, 2.5)$

Setup

Benchmarks:

- Funnelsort2 (binary basic mergers)
- Funnelsort4 (four-way basic mergers)
- Quicksort (GCC)
- Quicksort (Bentley & McIlroy)
- msort-c (cache-aware)
- msort-m (cache-aware)
- R-merge

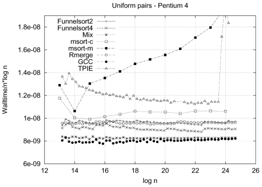
Workloads (RAM):

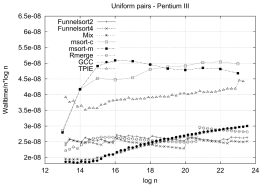
- Inputs of sizes in RAM range
- Median of 21 trials

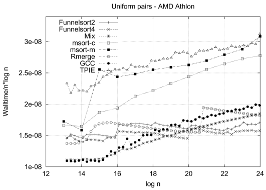
Workloads (Disk):

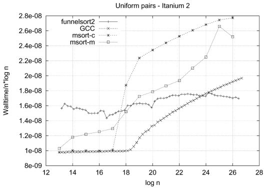
- Inputs on-disk
- Single-trial

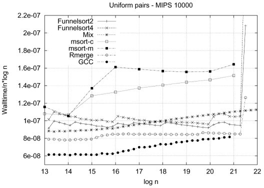
Data











Disk-based experiments omitted for brevity

Takeaways

1. Arbitrary memory layouts can still hold asymptotic properties
   - But vEB structure still has practical benefits
2. Iterative optimizations can lead to a competitive algorithm
3. Cache-obliviousness overhead can be worth it!

References

Gerth Stølting Brodal and Rolf Fagerberg.
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SIAM Journal on Computing, 1(1):31–39, 1972.

Donald E Knuth.
The art of computer programming, volume 3: Searching and sorting.
Addison-Westley Publishing Company: Reading, MA, 1973.

"Constructive feedback"

- Graphs
  - Too dense and somewhat poorly organized
  - Legend labels are inconsistent between graphs
- "std::sort() is really good" isn't very novel
- Memory layout observation is exciting, but is eventually disappointing
- Methodology and experiment setups are fairly detailed and precise
- Engineering phase pattern is useful
  - Though visuals (e.g. graphs) would have been helpful

Discussion

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