Paper Presentation

High-Performance Parallel Graph Coloring with Strong Guarantees on Work, Depth, and Quality

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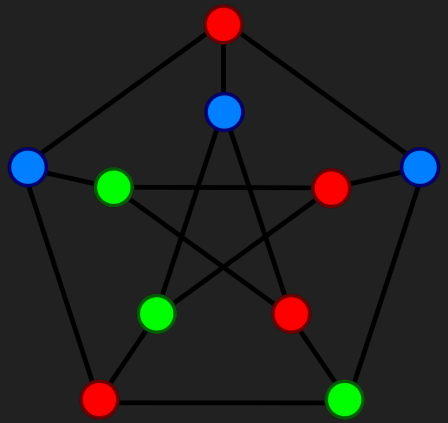
Graph Coloring



- Also known as *vertex coloring*
- ***vertex coloring***: an assignment of vertices where no two neighboring vertices share the same color
- ***k-coloring***: a vertex coloring of a graph using k distinct colors
- ***chromatic number (χ(G))***: the lowest number of colors k that yields a successful k-coloring
- ***optimal coloring (or coloring problem)***: the problem of finding χ(G) given a graph G
- optimal coloring is an NP-complete problem

Heuristics

- Used to compute colorings
- ***greedy heuristic***: for all colors, choose the smallest color not already chosen by a neighbor for vertex v
  - Gives a guarantee for coloring of a graph G with at most $\Delta + 1$ colors, where Δ is the maximum degree of the graph
- ***vertex ordering heuristic***: a heuristic which decides in which order the vertices are colored. Some examples are
  - ***first fit (FF)***: coloring via the default order in the vertices
  - ***largest-degree-first (LF)***: orders vertices based on degree in descending order
  - ***random (R)***: chooses vertices uniformly at random
  - ***incidence-degree (ID)***: largest number of uncolored neighbors first
  - ***saturation-degree (SD)***: largest number of distinct colors used by neighbors first
  - ***smallest-degree-last (SL)***: removes lowest degree vertices, colors remaining graph, colors removed vertices
- Sadly, with all of these combined with greedy, these algorithms have **no parallelism**

Parallel Graph Coloring Algorithms

- Jones and Plassman combined past heuristic work with parallel maximum independent set deriving algorithms to make ***JP***, a parallel graph coloring algorithm
  - Colors a vertex once all of its neighbors that come later in the ordering have been colored
- JP combined with random vertex ordering (***JP-R***) ran with O(n+m) work, O(log n/ log log n) depth for constant degree graphs (n = number of vertices, m = number of edges)
- Hasenplaugh et. al. use approximate LF and SL to create ***JP-LLF*** and ***JP-SLL***, which improve upon work and depth bounds over JP-R
- Another approximate SL ordering was used to create ***JP-ASL***, but offers no work or depth bounds (scary)

Problems With Aforementioned Algorithms

- None have strong guarantees of quality!
  - Quality: the closeness of a result to the chromatic number (true optimal coloring)
  - JP-R may yield poor quality
  - JP-LF and JP-SL yield better quality, but can balloon in performance (run in $\Omega(n)$ or $\Omega(\Delta^2)$ in some instances)
  - JP-LLF and JP-SLL yield quality of JP-LF and JP-SL with performance within a logarithmic factor of JP-R
- None of the above algorithms have any upper bound coloring quality guarantees

What The Paper Offers

- The first graph coloring algorithms with provably good bounds on work, depth, and quality
- These algorithms do this using a new vertex ordering heuristic: ***provably approximate degeneracy ordering (ADG)***
- An algorithm using this heuristic with JP, called ***JP-ADG***, with proven strong bounds on work, depth and quality
- An algorithm using this ordering with speculative coloring, called ***DEC-ADG***, with proven strong bounds on work, depth and quality
- “A use case of how ADG can seamlessly enhance an existing state-of-the-art graph coloring scheme (***DEC-ADG-ITR***)”
- Extensive theoretical analysis of and experimentation with graph coloring algorithms
- Superior coloring quality in practice

Fundamental Concepts: Graph Model and Representation

G	A graph $G = (V, E)$; V and E are sets of vertices and edges.
$G[U]$	$G[U] = (U, E[U])$ is a subgraph of G induced on $U \subseteq V$.
n, m	Numbers of vertices and edges in G; $\|V\| = n, \|E\| = m$.
$\Delta, \delta, \widehat{\delta}$	Maximum degree, minimum degree, and average degree of G.
d	The degeneracy of G.
$deg(v), N(v)$	The degree and the neighborhood of a vertex $v \in V$.
$deg_U(v)$	The degree of v in a subgraph induced by the vertex set $U \subseteq V$.
$N_U(v)$	The neighborhood of v in a subgraph induced by $U \subseteq V$.
$\rho_X(v)$	A priority function $V \to \mathbb{R}$ associated with vertex ordering X.
P	The number of processors (in a given PRAM machine).

TABLE I: Selected symbols used in the paper. When we use a symbol in the context of a specific loop iteration ℓ, we add ℓ in brackets or as subscript (e.g., $\widehat{\delta}_\ell$ is $\widehat{\delta}$ in iteration ℓ).

Fundamental Concepts: Degeneracy, Coreness, and Related Concepts

- ***s-degenerate***: a graph is s-degenerate if, in each of its induced subgraphs, there is a vertex with a degree of at most s.
- ***degeneracy***: the degeneracy d of a graph is the smallest s where the graph is still s-degenerate
- ***degeneracy ordering***: an ordering where there are at most d neighbors of each vertex ordered higher than it
- ***k-approximate degeneracy ordering***: same as exact, but at most k * d neighbors
- ***partial k-approximate degeneracy ordering***: same as k-approximate, but certain vertices can be ranked equally
- ***k-core***: A connected component that is left over after iteratively removing vertices with degree less than k from G
- ***coreness***: the largest possible k, such that v is part of a subgraph S of G with minimum degree k

Fundamental Concepts: Models For Algorithm Analysis

- Compute model: DAG model of dynamic multithreading
- Machine model: Ideal parallel computer
- Work-Depth Analysis for bounding runtimes
  - Uses both concurrent reads exclusive writes (CREW) and concurrent reads concurrent writes (CRCW)

Fundamental Concepts: Compute Primitives

- ***Reduce***
  - Given a set S, it finds and returns the sum of the set in O(n) work and O(log n) depth
  - Used to implement Count(S), which simply returns the size of S
- ***DecrementAndFetch (DAF)***
  - Atomically decrements its operand and returns a new value
  - Used to implement Join to synchronize processors

Fundamental Concepts: Randomization

- ***Monte Carlo Algorithms***: algorithms which return the correct result w.h.p
- ***Las Vegas Algorithms***: algorithms that always return the correct answer but have probabilistic runtime bounds
  - JP is a Las Vegas algorithm
- ***Coupling***: A coupling of two random variables X and Y is defined as a new variable (X' , Y') over the joint probability space, such that the marginal distribution of X' and Y' coincides with the distribution of X and Y respectively

Parallel Approximate Degeneracy Ordering

```
1  /* Input: A graph G(V, E).
2   * Output: A priority (ordering) function ρ : V → ℝ. */
3
4  D = [ deg(v_1) deg(v_2) ... deg(v_n) ] //An array with vertex degrees
5  ℓ = 1; U = V //U is the induced subgraph used in each iteration ℓ
6
7  while U ≠ ∅ do:
8    |U| = Count(U); //Derive |U| using a primitive Count
9    cnt = Reduce(U); //Derive the sum of degrees in U: ∑_{v∈U} D[v]
10   δ̂ = cnt/|U| //Derive the average degree for vertices in U
11
12   //R contains vertices assigned priority in a given iteration:
13   R = {u ∈ U | D[u] ≤ (1 + ε)δ̂ }
14   UPDATE(U, R, D) //Update D to reflect removing R from U
15   U = U \ R //Remove selected low-degree vertices (that are in R)
16   for all v ∈ R do in parallel: //Set the priority of vertices
17     ρ_ADG(v) = ℓ //The priority is the current iteration number ℓ
18   ℓ = ℓ + 1
19
20 //Update D to reflect removing vertices in R from a set U:
21 UPDATE(U, R, D):
22 for all v ∈ R do in parallel:
23   for all u ∈ N_U(v) do in parallel:
24     DecrementAndFetch(D[u])
```

Algorithm 1: **ADG**, our algorithm for computing the $2(1+\varepsilon)$-approximate degeneracy ordering; it runs in the CRCW setting.

ε here is an approximation accuracy term

Parallel Approximate Degeneracy Ordering: Design Details

- D is an array
- U and R are n-bit dense bitmaps
  - Updates and contains() functions run in O(1) time
  - Constructing R takes O(1) depth and O(|U|) work
  - U = U\R takes O(1) depth and O(|R|) work
  - Average degree calculation takes O(log n) depth and O(|U|) work

Parallel Approximate Degeneracy Ordering: Depth

- O(log n) depth of contents in the while loop
- While loop runs O(log n) iterations
- Overall depth: $O(\log^2 n)$

Lemma 1. *For a constant $\varepsilon > 0$, ADG does $O(\log n)$ iterations and has $O(\log^2 n)$ depth in the CRCW setting.*

Proof. At each step ℓ of the algorithm we can have at most $\frac{n}{1+\varepsilon}$ vertices with a degree larger than $(1+\varepsilon)\widehat{\delta}_\ell$. This can be seen from the fact, that the sum of degrees in the current subgraph can be at most n times the average degree $\widehat{\delta}_\ell$. For vertices with a degree exactly $(1+\varepsilon)\widehat{\delta}_\ell$ we get $\frac{n}{1+\varepsilon} \cdot (1+\varepsilon)\widehat{\delta}_\ell = n\widehat{\delta}_\ell$, which would result in a contradiction if we had more than $\frac{n}{1+\varepsilon}$ vertices with larger degree. Thus, if we remove all vertices with degree $\leq (1+\varepsilon)\widehat{\delta}_\ell$, *we remove a constant fraction of vertices in each iteration* (at least $\frac{\varepsilon}{1+\varepsilon}n$ vertices), which implies that ADG performs $O(\log n)$ iterations in the worst case, immediately giving the $O(\log^2 n)$ depth. To see this explicitly, one can define a simple recurrence relation for the number of iterations $T(n) \leq 1 + T\left(\frac{1}{1+\varepsilon}n\right)$, $T(1) = 1$; solving it gives $T(n) \leq \left\lceil \frac{\log n}{\log(1+\varepsilon)} + 1 \right\rceil \in O(\log n)$. □

Parallel Approximate Degeneracy Ordering: Work

- Each iteration of the while loop does O((deg v for all v in R) + U_i) work.
- (deg v for all v in R) for all i = 1 to k = O(m)
- U_i from i=1 to k can be bound by a geometric series, implying that it is still O(n)
- Overall runtime: O(m + n)

Work The proof of work is similar; it also uses the fact that a constant fraction of vertices is removed in each iteration. Intuitively, (1) we show that each `while` loop iteration performs $O\left(\left(\sum_{v \in R} deg(v)\right) + |U_i|\right)$ work (where U_i is the set U in iteration i), and (2) we bound $\sum_{i=1}^{k} |U_i|$ by a geometric series, implying that it is still in $O(n)$.

Lemma 2. *For a constant $\varepsilon > 0$, ADG does $O(n+m)$ work in the CRCW setting.*

Proof. Let k be the number of iterations we perform and let U_i be the set U in iteration i. To calculate the total work performed by ADG, we first consider the work in one iteration. As explained in "Design Details", deriving the average degree takes $O(|U_i|)$ work in one iteration. Initializing R takes $O(|U_i|)$ and removing R from U_i takes $O(|R|)$. UPDATE takes $O\left(\sum_{v \in R} deg(v)\right)$ work. Thus, the total work in one iteration is in $O\left(\left(\sum_{v \in R} deg(v)\right) + |U_i|\right)$. As each vertex becomes included in R in a single unique iteration, this gives $\sum_{i=1}^{k} \sum_{v \in R_i} deg(v) \in O(m)$. Moreover, since we remove a constant number of vertices in each iteration from U (at least $\frac{\varepsilon}{1+\varepsilon}$ as shown above in the proof of the depth of ADG), we can bound $\sum_{i=1}^{k} |U_i|$ by a geometric series, implying that it is still in $O(n)$. This can be seen from the fact that $\sum_{i=1}^{k} |U_i| \leq \sum_{i=0}^{k} \left(\frac{1}{1+\varepsilon}\right)^i n \leq \sum_{i=0}^{\infty} \left(\frac{1}{1+\varepsilon}\right)^i n = \frac{(1+\varepsilon)}{\varepsilon} n$ if $\frac{1}{1+\varepsilon} < 1$ (which holds as $\varepsilon > 0$). Ultimately, we have $O\left(\sum_{i=0}^{k} \left(\sum_{v \in R_i} deg(v)\right) + |U_i|\right) \in O(m) + O(n) \in O(m+n)$. □

Parallel Approximate Degeneracy Ordering: Quality

- Approximation ratio of 2(1+ε)
- All vertices removed in a step have a degree of at most (1+ε)*the average degree of vertices in the subgraph at that point, which is upper bounded by 2d
- Therefore, each vertex has at most 2(1+ε)d neighbors that are ranked equal or higher.

Approximation ratio We now prove that the approximation ratio of ADG on the degeneracy order is $2(1+\varepsilon)$. First, we give a small lemma used throughout the analysis.

Lemma 3. *Every induced subgraph of a graph G with degeneracy d, has an average degree of at most $2d$.*

Proof. By the definition of a d-degenerate graph, in every induced subgraph $G[U]$, there is a vertex v with $deg_U(v) \leq d$. If we remove v from $G[U]$, at most d edges are removed. Thus, if we iteratively remove such vertices from $G[U]$, until only one vertex is left, we remove at most $d \cdot (|U| - 1)$ edges. We conclude that $\widehat{\delta}(G[U]) = \frac{1}{|U|} \sum_{v \in U} deg_U(v) \leq 2d$. □

Lemma 4. *ADG computes a partial $2(1+\varepsilon)$-approximate degeneracy ordering of G.*

Proof. By the definition of R (Line 13), all vertices removed in step ℓ have a degree of at most $(1+\varepsilon)\widehat{\delta}_\ell$, where $\widehat{\delta}_\ell$ is the average degree of vertices in subgraph U in step ℓ. From Lemma 3, we know that $\widehat{\delta}_\ell \leq 2d$. Thus, each vertex has a degree of at most $2(1+\varepsilon)d$ in the subgraph $G[U]$ (in the current step). Hence, each vertex has at most $2(1+\varepsilon)d$ neighbors that are ranked equal or higher. The result follows by the definition of a partial $2(1+\varepsilon)$-approximate degeneracy order. □

Parallel Approximate Degeneracy Ordering: Extras

- Graph to the right shows ADG compared to other ordering heuristics
- In the CREW model, depth stays the same, while work increases to O(m+nd), after redesigning the update function

Ordering heuristic	Time / Depth	Work	F.?	B., Approx.?
FF (first-fit) [25]	$O(1)$	$O(1)$	n/a	n/a
R (random) [26], [31]	$O(1)$	$O(n)$	👍	n/a
ID (incidence-degree) [1]	$O(n+m)$	$O(n+m)$	n/a	n/a
SD (saturation-degree) [27], [31]	$O(n+m)$	$O(n+m)$	n/a	n/a
LF (largest-degree-first) [31]	$O(1)$	$O(n)$	👍	n/a
LLF (largest-log-degree-first) [31]	$O(1)$	$O(n)$	👍	n/a
SLL (smallest-log-degree-last) [31]	$O(\log \Delta \log n)$	$O(n+m)$	👍	👎
SL (smallest-degree-last) [28], [31]	$O(n)$	$O(m)$	👍	👍 exact
ASL (approximate-SL) [32]	$O(n)$	$O(m)$	👍	👎
ADG [approx. degeneracy]	$O\left(\log^2 n\right)$	$O(n+m)$	👍	👍 $2(1+\varepsilon)$

TABLE II: **Ordering heuristics related to the degeneracy ordering. "F. (Free)?":** Is the scheme free from concurrent writes? **"B. (Bounds)?":** Are there **provable** bounds and approximation ratio on degeneracy ordering? "👍": support, "👎": no support. Notation is explained in Table I and in Section II.

```
1 UPDATE(U_i, R_i, D): //U_i and R_i are sets U and R in iteration i
2   for all v ∈ U_i do in parallel:
3     D[v] = D[v] − Count(N_{U_i}(v) ∩ R_i)
```

Algorithm 2: The modified version of the UPDATE routine from **ADG** (Algorithm 1) that works in the CREW setting.

JP-ADG

```
1  /* Input: A graph G(V, E), a priority function ρ.
2   * Output: An array C, it assigns a color to each vertex. */
3
4  //Part 1: compute the DAG G_ρ based on ρ
5  C = [0 0 ... 0] //Initialize colors
6  for all v ∈ V do in parallel:
7    //Predecessors and successors of each vertex v in G_ρ:
8    pred[v] = {u ∈ N(v) | ρ(u) > ρ(v)}
9    succ[v] = {u ∈ N(v) | ρ(u) < ρ(v)}
10   //Number of uncolored predecessors of each vertex v in G_ρ:
11   count[v] = |pred[v]|
12
13 //Part 2: color vertices using G_ρ
14 for all v ∈ V do in parallel:
15   //Start by coloring all vertices without predecessors:
16   if pred[v] == ∅: JPColor(v)
17
18 JPColor(v) //JPColor, a routine used in JP
19   C[v] = GetColor(v)
20   for all u ∈ succ[v] in parallel:
21     //Decrement u's counter to reflect that v is now colored:
22     if Join(count[u]) == 0:
23       JPColor(u) //Color u if it has no uncolored predecessors
24
25 GetColor(v) //GetColor, a routine used in JPColor
26   C = {1, 2, ..., |pred[v]| + 1}
27   for all u ∈ pred[v] do in parallel: C = C − {C[u]}
28   return min(C) //Output: the smallest color available.
```

Algorithm 3: **JP**, the Jones-Plassman coloring heuristic. With $\rho = \langle \rho_{\text{ADG}}, \rho_{\text{R}} \rangle$, it gives **JP-ADG** that provides $(2(1+\varepsilon)d+1)$-coloring.

JP-ADG: Quality

Coloring Quality The coloring quality now follows from the properties of the priority function obtained with ADG.

Corollary 1. *With priorities* $\rho = \langle \rho_{ADG}, \rho_R \rangle$*, JP-ADG colors a graph with at most* $2(1+\varepsilon)d+1$ *colors, for* $\varepsilon > 0$.

Proof. Since $\langle \rho_{\text{ADG}}, \rho_R \rangle$ is a $2(1+\varepsilon)$-approximate degeneracy ordering, the result follows from Lemma 6 and 4. □

JP-ADG: Depth, Work

- Expected longest path in DAG G_ρ is $O(d \log n + \log d \log^2 n / \log \log n)$
- Expected depth is $O(\log^2 n + \log \Delta * (\text{longest path})$
- Work is O(n+m) since both ADG and JP take O(n+m) work

Theorem 1. *JP-ADG colors a graph G with degeneracy d in expected depth $O(\log^2 n + \log \Delta \cdot (d \log n + \frac{\log d \log^2 n}{\log \log n}))$ and $O(n+m)$ work in the CRCW setting.*

Proof. Since ρ_{ADG} is a partial $2(1+\varepsilon)$-approximate degeneracy ordering (Lemma 4) and since ADG performs at most $O(\log n)$ iterations (Lemma 1), the depth follows from Lemma 7, Lemma 1 and past work [31], which shows that JP runs in $O(\log n + \log \Delta \cdot |\mathcal{P}|)$ depth. As both JP and ADG perform $O(n+m)$ work, so does JP-ADG. □

Proof. Let $G_\rho(\ell)$ be the subgraph of G_ρ induced by the vertex set $V(\ell) = \{v \in V \mid \rho_{\text{ADG}} = \ell\}$. Let Δ_ℓ be the maximal degree and $\widehat{\delta}_\ell$ be the average degree of the subgraph $G_\rho(\ell)$.

Since, by the definition of G_ρ, there can be no edges in G_ρ that go from one subgraph $G_\rho(\ell)$ to another $G_\rho(\ell')$ with $\ell' > \ell$, we can see that a longest (directed) path $\mathcal{P}$ in G_ρ will always go through the subgraph $G_\rho(\ell)$ in a monotonically decreasing order with regards to ℓ. Therefore, we can split $\mathcal{P}$ into a sequence of (directed) sub-paths $\mathcal{P}_1, \ldots, \mathcal{P}_{\overline{\rho}}$, where $\mathcal{P}_\ell$ is a path in $G(\ell)$. We have $|\mathcal{P}| = \sum_{i \in \{\rho_{\text{ADG}}(v) | v \in V\}} |\mathcal{P}_i|$ and by Corollary 6 from past work [31], the expected length of a longest sub-path $\mathcal{P}_\ell$ is in $O(\Delta_\ell + \log \Delta_\ell \log n / \log \log n)$, because $G_\rho(\ell)$ is induced by a random priority function. By linearity of expectation, we have for the whole path $\mathcal{P}$:

$$\mathbb{E}[|\mathcal{P}|] = O\left(\sum_{\ell=1}^{\overline{\rho}} \left(\Delta_\ell + \log \Delta_\ell \cdot \frac{\log n}{\log \log n}\right)\right) \quad (1)$$

Next, since ρ_{ADG} is a partial k-approximate degeneracy ordering, all vertices in $G(\ell)$ have at most kd neighbors in $G(\ell)$. Thus, $\Delta_\ell \leq kd$ holds. This and the fact that $\overline{\rho} \in O(\log n)$ gives:

$$\sum_{i=1}^{\overline{\rho}} \Delta_i \leq \sum_{i=1}^{\overline{\rho}} d \cdot k \in O(d \log n) \quad (2)$$

$$\sum_{i=1}^{\overline{\rho}} \log \Delta_i \in O(\log d \log n) \quad (3)$$

Thus, for the expected length of a longest path in G:

$$\mathbb{E}[|\mathcal{P}|] = O\left(d \log n + \frac{\log d \log^2 n}{\log \log n}\right) \quad (4)$$

□

Our main result follows by combining our bounds on the longest path $\mathcal{P}$ in the DAG G_ρ and a result by Hasenplaugh et al. [31], which shows that JP has $O(\log n + \log \Delta \cdot |\mathcal{P}|)$ depth.

DEC-ADG

```
/* Input: G(V, E) (input graph).
 * Output: An array C, it assigns a color to each vertex. */

C = [0 0 ... 0] //Initialize an array of colors
//Run ADG* to derive a 2(1 + ε/12)-approximate degeneracy ordering
//𝒢 ≡ [G(1) ... G(p̄)] contains p̄ low-degree partitions
//We have G(i) = G[R(i)] where R(i) = {v ∈ V | ρ(v) = i}
(ρ, 𝒢) = ADG*(G)

//Initialize bitmaps B_v to track colors forbidden for each vertex
∀_{v∈V} B_v = [00...0] //Each bitmap is ⌈2(1 + ε/12)(1 + μ)d⌉ + 1 bits
SIM-COL(G(p̄), {B_v | v ∈ R(p̄)}) //First, we color G(p̄)

for ℓ from p̄ − 1 down to 1 do: //For all low-degree partitions
  Q = R(p̄) ∪ ··· ∪ R(ℓ + 1) //A union of already colored partitions
  for all v ∈ R(ℓ) do in parallel:
    for all u ∈ N_Q(v) do in parallel: // For v's colored neighbors
      B_v = B_v ∪ C[u] // Update colors forbidden for v
  SIM-COL(G(ℓ), {B_v | v ∈ R(ℓ)}) //Run Algorithm 5
```

Algorithm 4: **DEC-ADG**, the second proposed parallel coloring heuristic that provides a $(2(1+\varepsilon)d)$-coloring. Note that we use factors $\varepsilon/4$ and $\varepsilon/12$ for more straightforward proofs (this is possible as ε can be an arbitrary non-negative value).

- Employs the use of a function called SIM-COL (Simple Coloring), which colors a subgraph using (1+μ)Δ colors

SIM-COL

```
1 /* Input: G(V(ℓ), E(ℓ)) (input graph partition), B_v (a bitmap with
2  * colors forbidden for each v). Output: color C[v] > 0. */
3 U = V(ℓ)
4 while U ≠ ∅ do:
5   //Part 1: all vertices in U are colored randomly:
6   for all v ∈ U do in parallel:
7     choose C[v] u.a.r. from {1, ..., (1 + μ)deg_ℓ(v)}
8
9   //Part 2: each vertex compares its color to its active neighbors.
10  //If the color is non-unique, the vertex must be re-colored.
11  for all v ∈ U do in parallel:
12    //f_eq(v, u) = (C[v] == C[u]) is the operator in Reduce.
13    if Reduce(N_U(v), f_eq) > 0 || C[v] ∈ B_v: C[v] = 0
14
15  //Part 3: Update B_v for all neighbors with fixed colors:
16  for all v ∈ U do in parallel:
17    for all u ∈ N_U(v) do in parallel:
18      if C[u] > 0: B_v = B_v ∪ C[u]
19  U = U \ {v ∈ U | C[v] > 0} //Update vertices still to be colored
```

Algorithm 5: **SIM-COL**, our simple coloring routine used by DEC-ADG. It delivers a $((1+\mu)\Delta)$-coloring, where $\mu > 0$ is an arbitrary value. When using SIM-COL as a subroutine in DEC-ADG, we instantiate μ as $\mu = \varepsilon/4$; we use this value in the listing above for concreteness.

SIM-COL: Work, Depth

- SIM-COL performs $O(\log n)$ iterations w.h.p
- The work in each iteration has depth $O(\Delta)$ in CREW and $O(\log \Delta)$ in CRCW
- Thus, depth is $O(\Delta \log n)$ and $O(\log \Delta \log n)$ in CREW and CRCW respectively
- Work is $O(n + m)$

DEC-ADG: Work, Depth, and Quality

- $O(\log d \log^2 n)$ depth in CRCW
- $O(n + m)$ work w.h.p
- Coloring quality: $(2+\varepsilon)d$ coloring of the graph for $0 < \varepsilon \leq 8$.
- Work and depth proofs of this algorithm and SIM-COL made possible by use of Markov Inequalities and Chernoff Bounds, taking advantage of randomness in conjunction with constants, to create high probability bounds on size and computation

DEC-ADG-ITR

- Same as DEC-ADG, but instead of choosing a random color in SIM-COL line 7, it chooses the smallest color not taken by a vertex v's neighbors

Selecting colors can be done in $O(\Delta)$ depth and $O(\Delta \cdot |U_i|)$ work, where U_i is the set U in iteration i (of the `while` loop in SIM-COL) and $\widetilde{\Delta} = \max_{v \in V(\ell)} deg_\ell(v)$. For the modified SIM-COL, we get, since all other operations are equal, $O(\Delta I)$ depth (I is the number of iterations of ITR); the work is

$$O\left(\sum_{i=1}^{I}\left[\widetilde{\Delta} \cdot |U_i| + \sum_{v \in U_i} deg(v)\right]\right).$$

Depth and work in DEC-ADG-ITER are, respectively

$$O(Id \log n)$$

and (as a simple sum over all iterations I)

$$O\left(\sum_{\ell=1}^{\overline{\rho}}\left(\sum_{i=1}^{I}\left[\widetilde{\Delta} \cdot |U_i| + \sum_{v \in U_i} deg(v)\right] + \sum_{v \in R(\ell)} deg(v)\right)\right).$$

Note that *these bounds are valid in the CREW setting*.

Using Concurrent Reads (CREW)

- Adds a (small) factor of d to a subcomponent of ADG, yielding a new work of $O(dn + m)$ for ADG and DEC-ADG

Optimizations

- Representation of U and R
  - R should be listed in the same array as U, where R(.) precedes U
  - Done so “the actual removal of R(i + 1) from U takes O(1) time by simply moving the index pointer by |R(i+1)| positions “to the right”, giving – at the end of iteration i + 1 – a representation [R(0) R(1) ... R(i) R(i + 1) index U]”
- Explicit Ordering in R(.)
  - “ often enhances the accuracy of the obtained approximate degeneracy ordering, which in turn consistently improves the ultimate coloring accuracy”
- Combining JP and ADG
  - “where one derives predecessors and successors in a given ordering to construct the DAG Gρ [in JP], can also be implemented as a part of UPDATE in ADG, in Algorithm 1”
  - Doesn’t improve theoretical results, but saves time in implementation

Optimizations Pt. 2

- Usage of degree median instead of degree average
  - Takes O(1) time in a sorted array
  - Variants with “-M” at the end use this variation
  - “However, the whole U has to be sorted in each pass. We incorporate linear-time integer sorting, which was shown to be fast in the context of sorting vertex IDs [86] ADG-M only differs from Algorithm 1 in that”
  - “Additionally we limit R to half the size of U (+1 if |U| is odd). We refer to the priority function produced by ADG-M as ρADG-M”
- Push vs Pull style updating
  - “ (pushing updates to a shared state or pulling updates to a private state)”
  - While pushing needs atomics, pulling takes more updates
  - Similar performance in practice
- Caching Sums of Degrees
  - When computing average degree, one can just subtract edges from existing sum instead of fully recomputing from all edges
  - Slightly improves runtime (~1%)

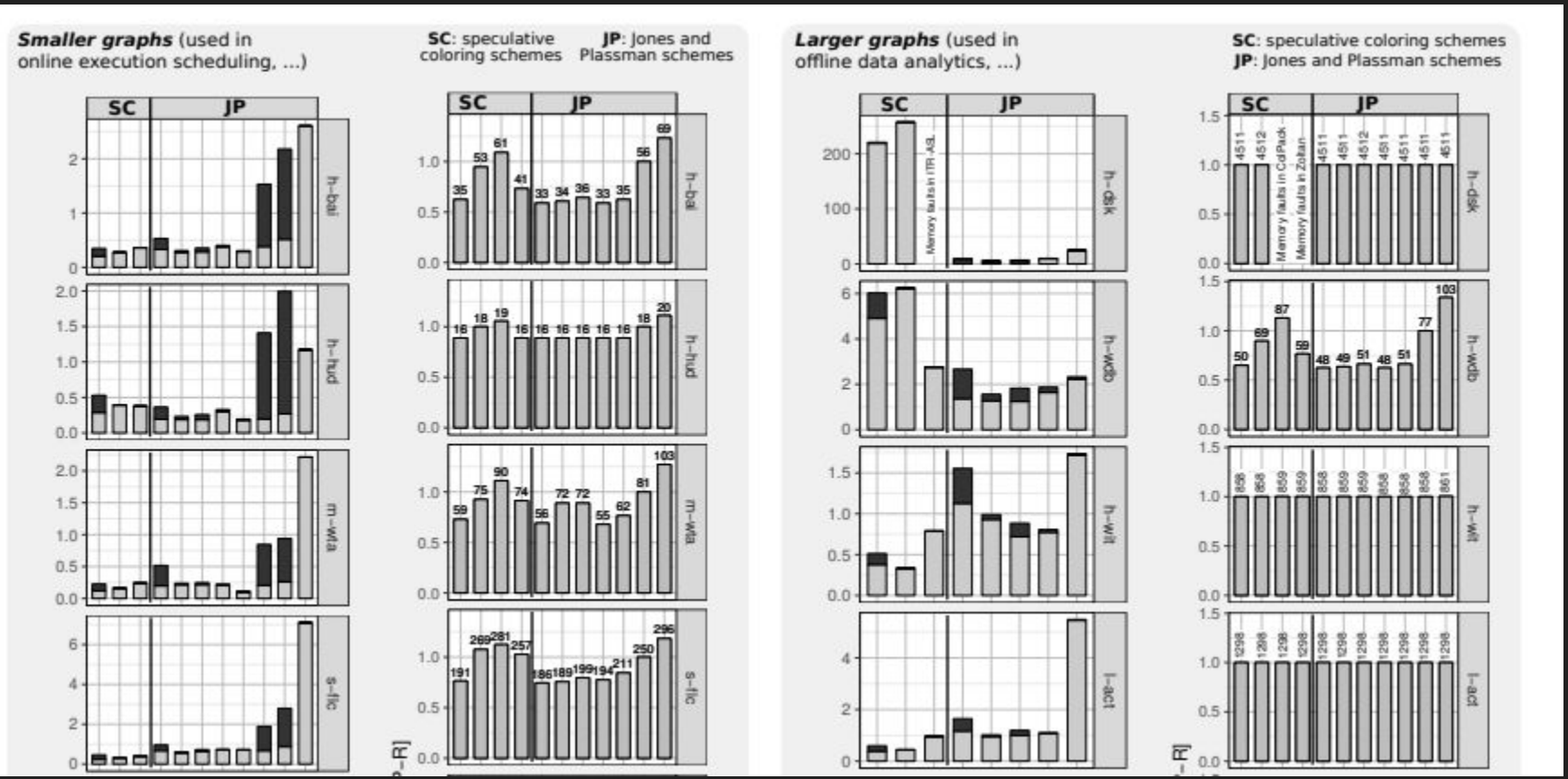
Implementation

- Integrated with GBBS and GAP benchmark suite
- Implemented with optimizations
- ADG-M
  - Work: $O(n + m)$
  - Depth: $O(\log^2 n)$, from half of the vertices being removed each iteration, and each iteration taking $O(\log n)$ work
  - ADG-M computes a partial 4-approximate degeneracy ordering of G
- JP-ADG-M
  - Same work and depth as JP-ADG

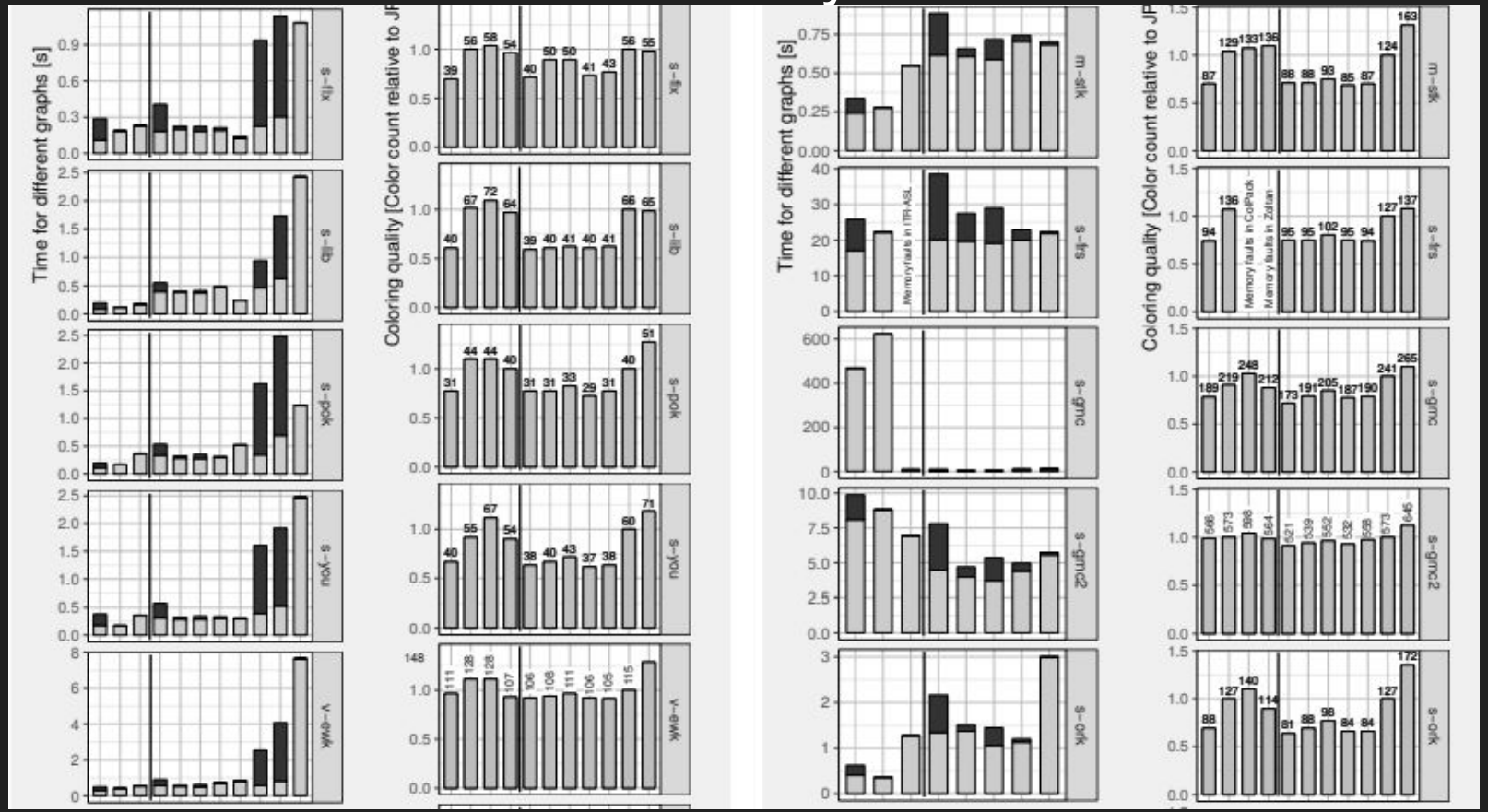
Evaluations

- Coloring Quality is superior
- Algorithm runtime is comparable or marginally higher
- “Thus, we offer the best coloring quality at the smallest required run-time overhead. Finally, our routines are the only ones with theoretical guarantees on work, depth, and quality.”

Evaluations: Runtime and Quality



Evaluations: Runtime and Quality



Evaluations: Runtime and Quality

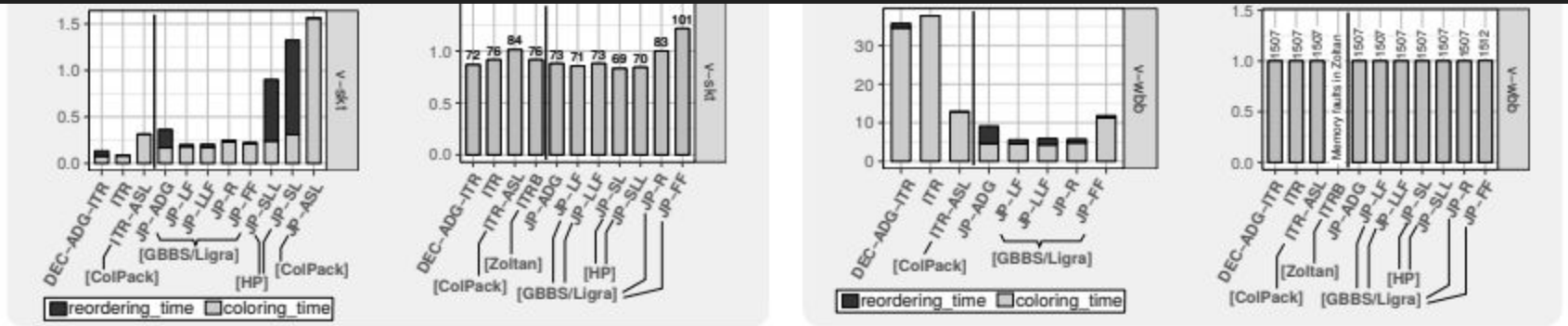


Fig. 1: **Run-times (1st and 3rd columns) and coloring quality (2nd and 4th columns)**. Two plots next to each other correspond to the same graph. Graphs are representative (other results follow similar patterns). Parametrization: 32 cores (all available), $\varepsilon = 0.01$, sorting: Radix sort, direction-optimization: push, JP-ADG variant based on average degrees $\widehat{\delta}$. SL and SLL are excluded from run-times in the right column (for larger graphs) because they performed consistently worse than others. We exclude DEC-ADG for similar reasons and because it is of mostly theoretical interest; instead, we focus on DEC-ADG-ITR, which is based on core design ideas in DEC-ADG. Numbers in bars for color counts are numbers of used colors. **"SC"**: results for the class of algorithms based on **speculative coloring** (ITR, DEC-ADG-ITR). **"JP"**: results for the class of algorithms based on the **Jones and Plassman** approach (**color scheduling**, JP-*). A vertical line in each plot helps to separate these two classes of algorithms. DEC-ADG-ITR uses dynamic scheduling. JP-ADG uses linear time sorting of R. Any schemes that are always outperformed in a given respect (e.g., Zoltain in runtimes or ColPack in qualities) are excluded from the plots.

Evaluations: Scaling

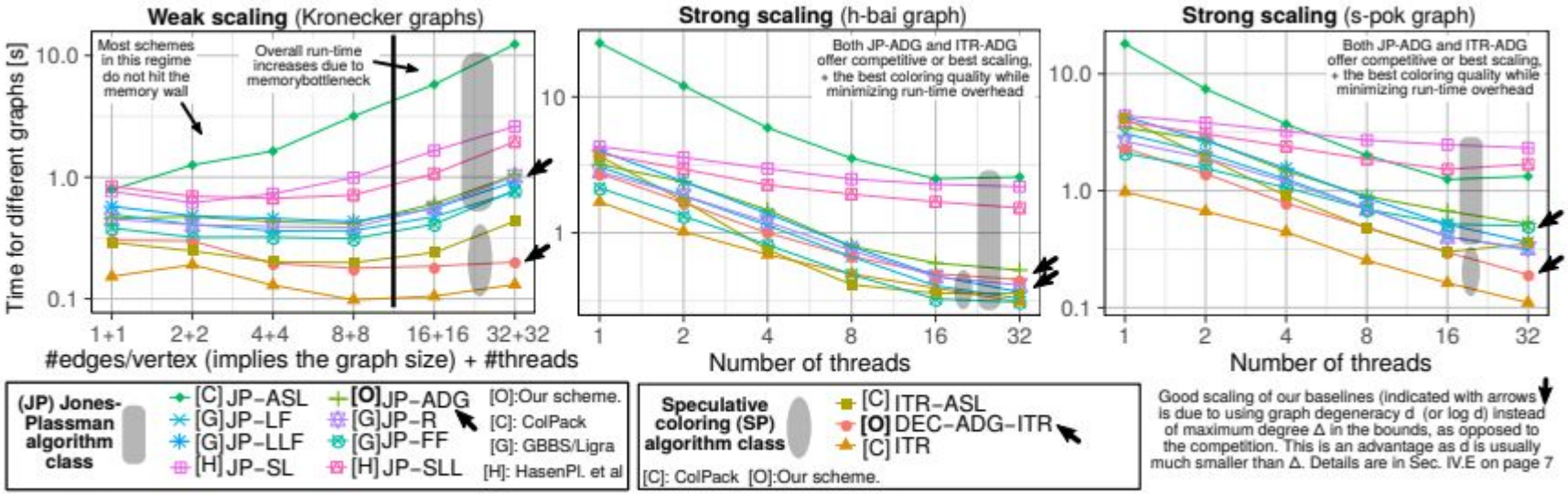


Fig. 2: **Weak and strong scaling**. Graphs are representative (other results follow similar patterns). Parametrization: $\varepsilon = 0.01$, sorting: Radix sort, direction-optimization: push, JP-ADG variant based on average degrees $\widehat{\delta}$. DEC-ADG-ITR uses dynamic scheduling. JP-ADG uses linear time sorting of R. In weak scaling, we use $n = 1$M vertices.

Evaluations: Impact of ε

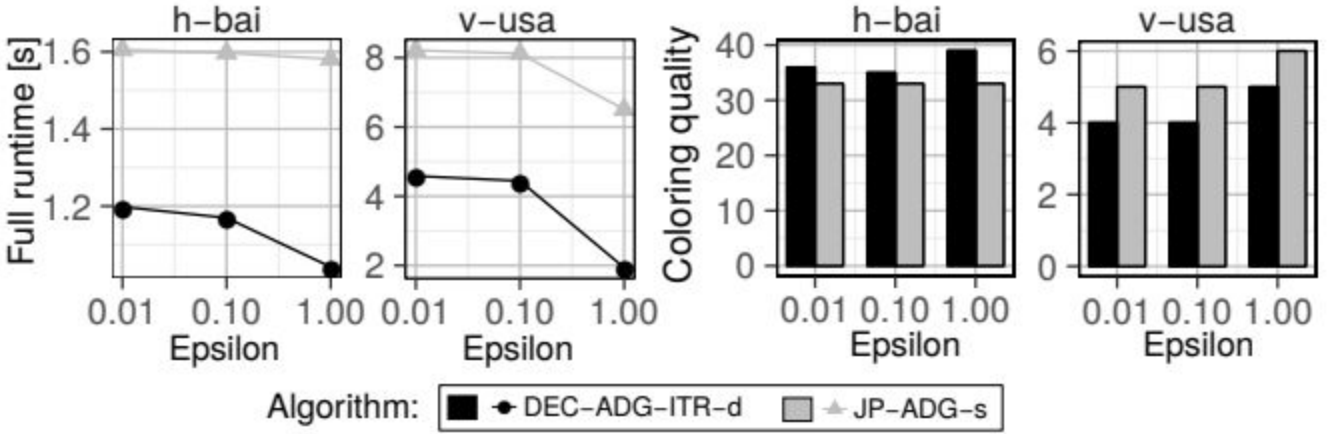


Fig. 3: **Impact of ε on run-times and coloring quality**. Parametrization: 32 cores, sorting: Radix sort, direction-optimization: push, JP-ADG variant based on average degrees $\widehat{\delta}$. DEC-ADG-ITR uses dynamic scheduling.

Evaluations: Memory Pressure

H. Memory Pressure and Idle Cycles

We also investigate the pressure on the memory bus, see Figure 4. For this, we use PAPI [102] to gather data about idle CPU cycles and L3 cache misses. Low ratios of L3 misses or idle cycles indicate high locality and low pressure on the memory bus. Overall, our routines have comparable or best ratios of active cycles and L3 hits.

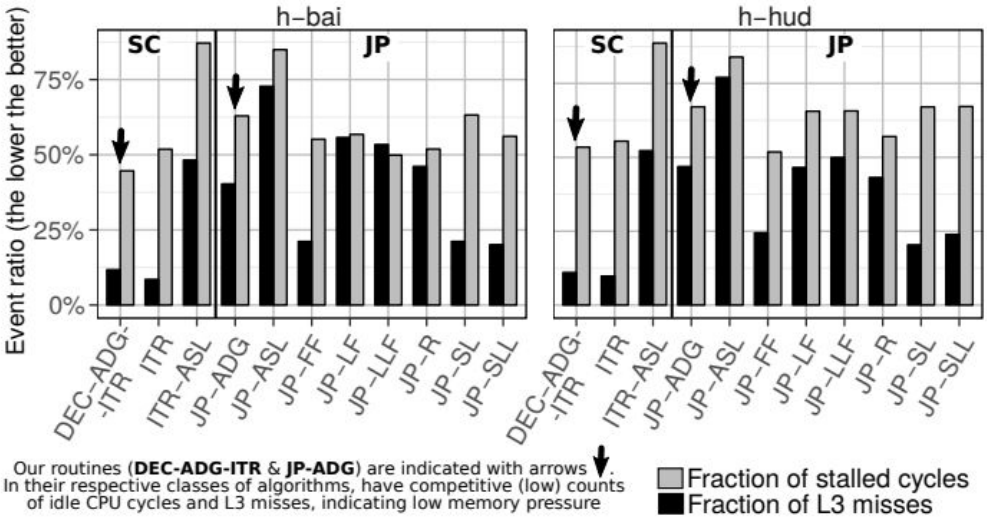


Fig. 4: **Fractions of L3 misses (out of all L3 accesses) and idle (stalled) CPU cycles (out of all CPU cycles) in each algorithm execution**. Parametrization: graph h-hud, 32 cores, sorting: Radix sort, direction-optimization: push, JP-ADG uses average degrees $\widehat{\delta}$. DEC-ADG-ITR uses dynamic scheduling. JP-ADG uses linear time sorting of R.

Evaluations: Color Quality Frequency

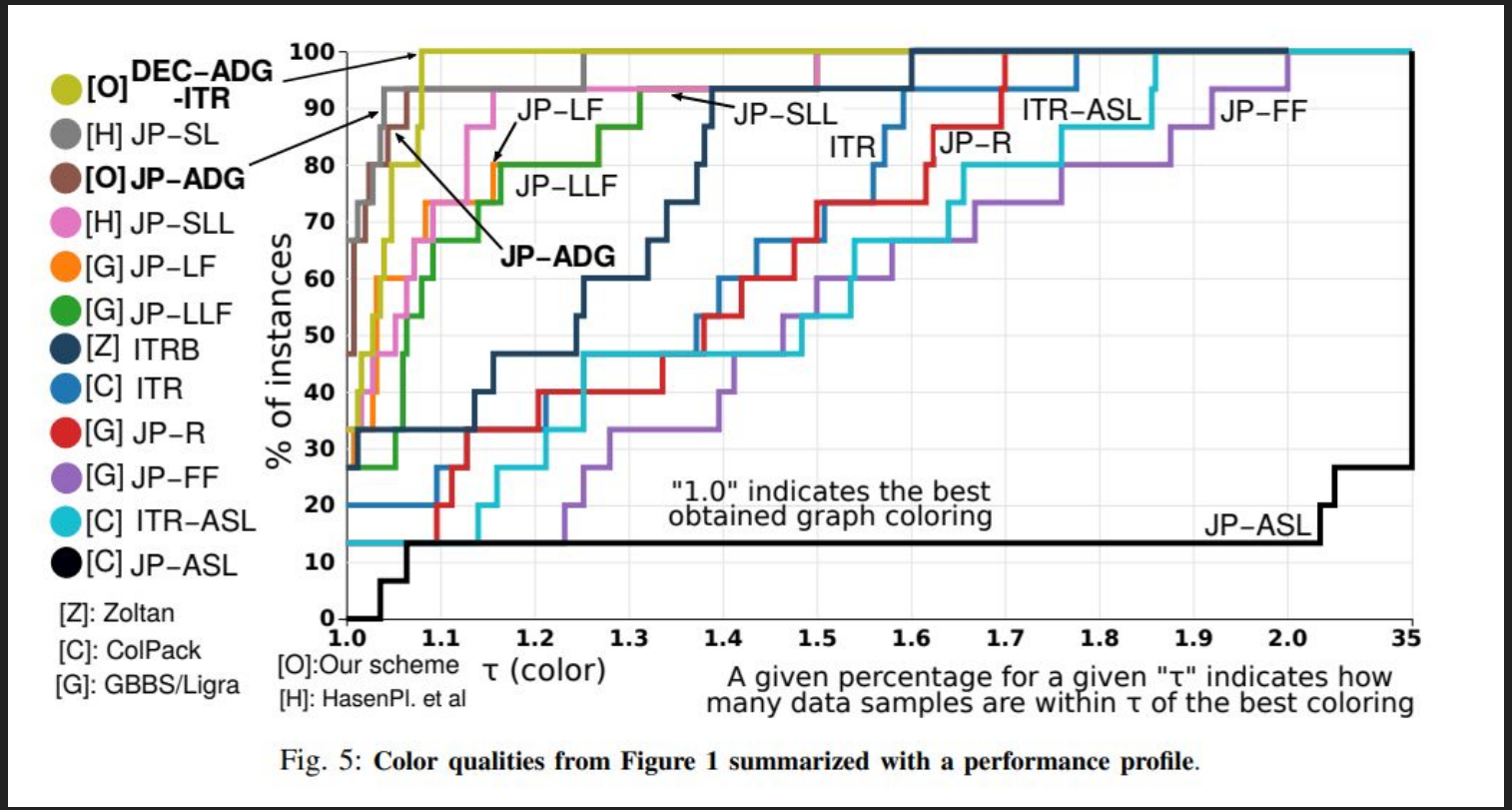


Fig. 5: **Color qualities from Figure 1 summarized with a performance profile.**

Thoughts

- Overall very strong paper
- Provides ground breaking results, explains concepts well, provided reader knowledge from ground up, thorough experimentation
- Nothing for me to really nitpick after reading

Discussion Questions

- Other practical optimizations that can be made? Theoretical?
- Is recoloring worthwhile?
- Applications?
- Potential approaches for exact parallel colorings?

Bibliography

- Image on slide 2 is from wikipedia’s graph coloring page https://en.wikipedia.org/wiki/Graph_coloring
- All other images come from the reviewed paper