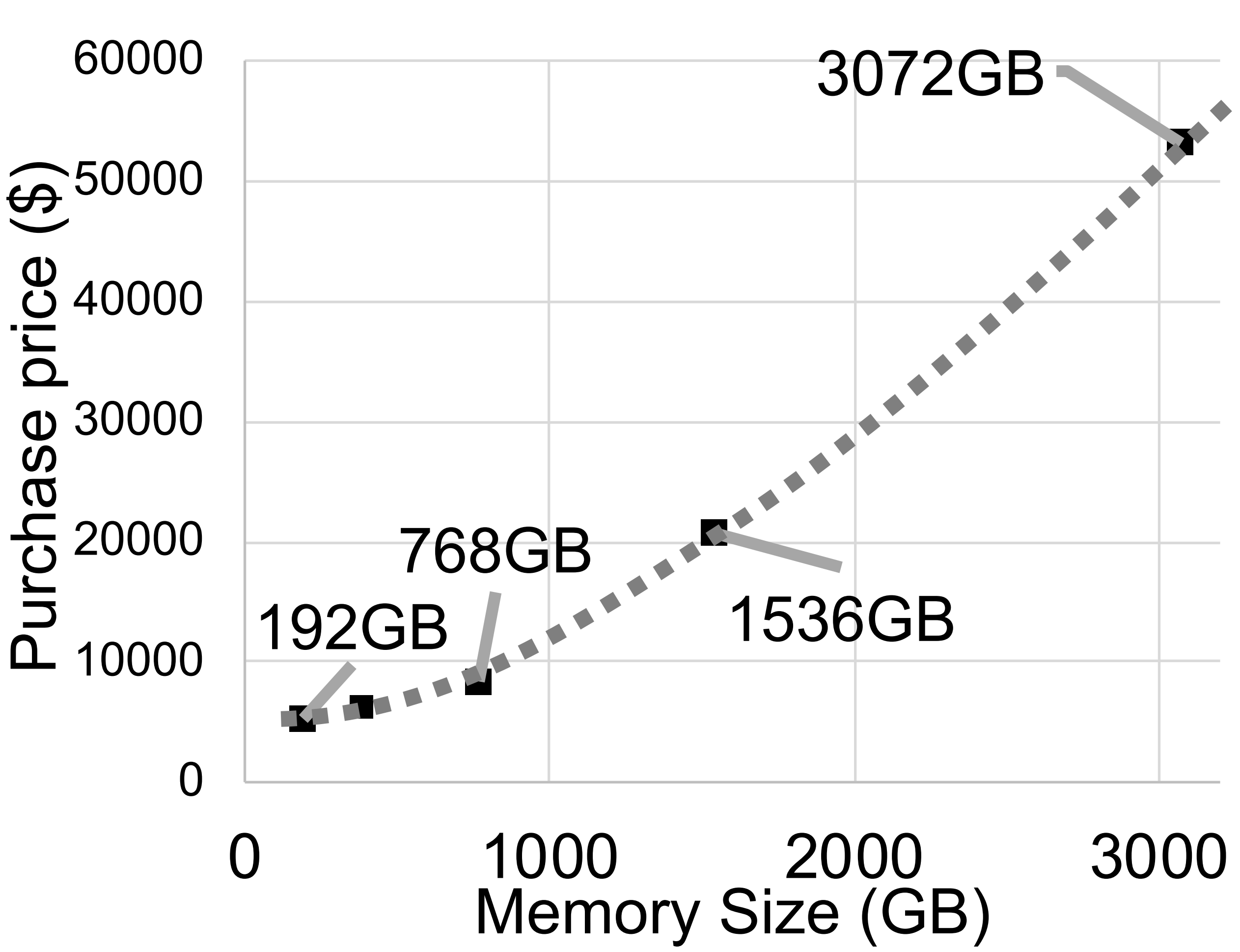
Parallel In-Place Algorithms: Theory and Practice

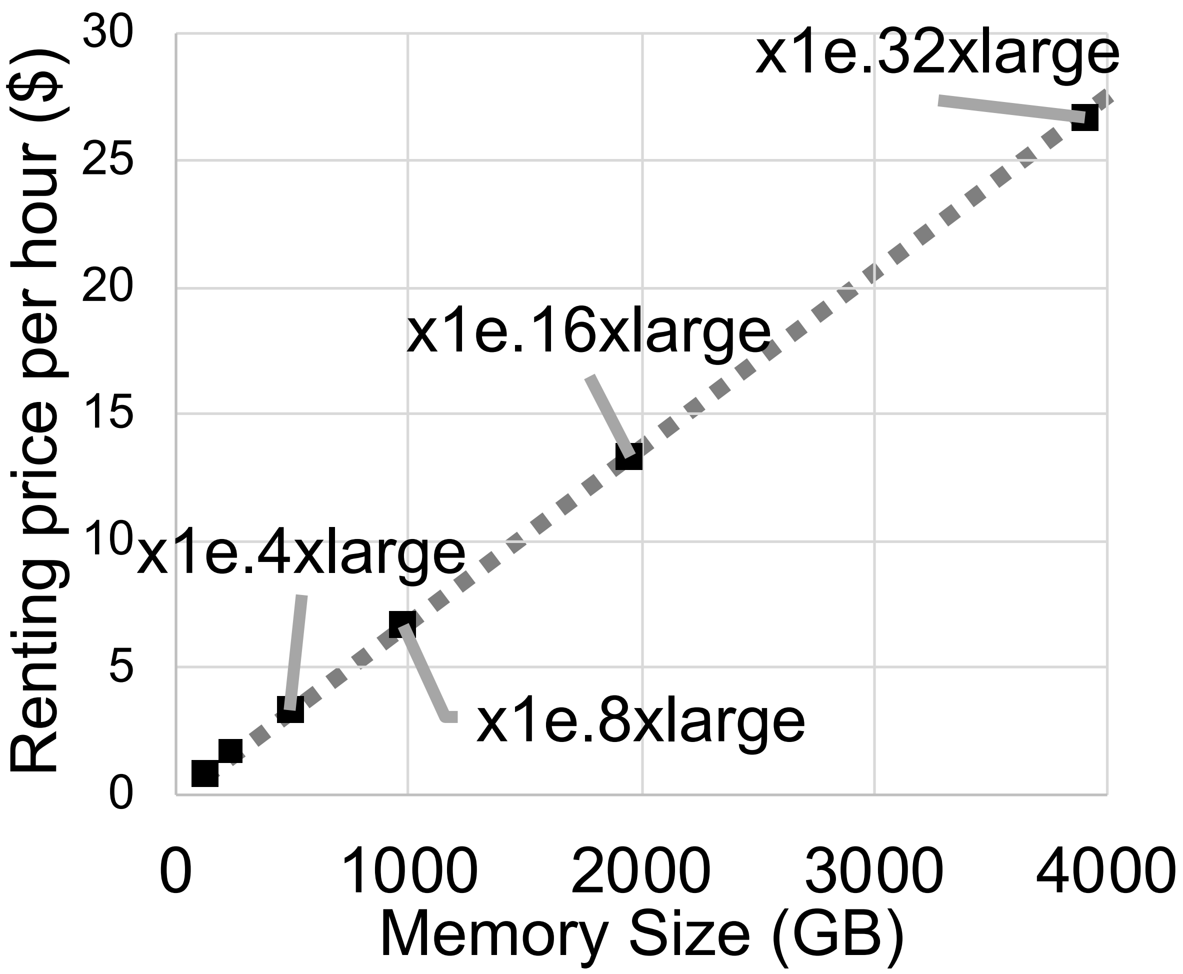
Yan Gu, Omar Obeya, and Julian Shun

(Based on slides by Yan Gu)

You can put more memory on a machine, but they are expensive



Purchase price of RAX XT24-42S1 with 72 CPU cores (Xeon Gold 5220)



Rental price of AWS EC2 x1e-series multicore instances

Space-efficiency is crucial for shared-memory parallel algorithms

- Allows you to run larger inputs on your machine
- Decreases monetary costs
- Reducing memory footprint can improve performance due to lower memory traffic and better cache utilization

Parallel in-place algorithms have been gaining attention recently, but they are still underexplored

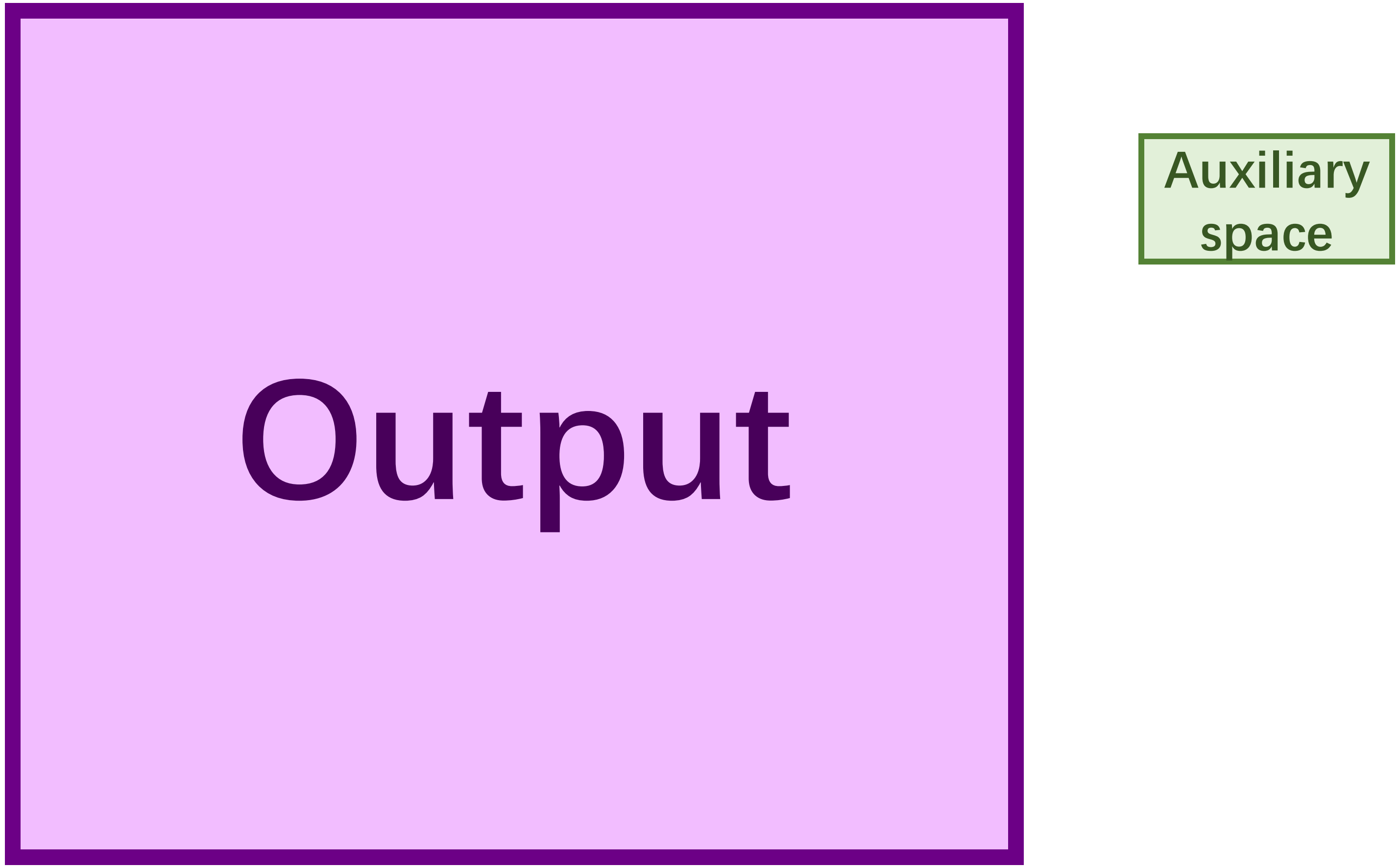
- Duplicate removing [HL89]
- Merge and mergesort [GL91, GL92]
- Samplesort [ZCZ99, AWFS17]
- Search problems (backtrack and branch-and-bound) [PPSV15]
- Generating search tree layout [BCH+18]
- Radix sort [OKFS19]
- Partition [KW20]

- Yet, there are no standard definition on what "parallel in-place" means
- Yet, there are no general approaches to designing parallel in-place algorithms

Outline of this talk

1. Models for parallel in-place (PIP) algorithms
2. New PIP algorithms and a general approach

In-place in the sequential setting



$$O(1)$$

$$O(\log n)$$

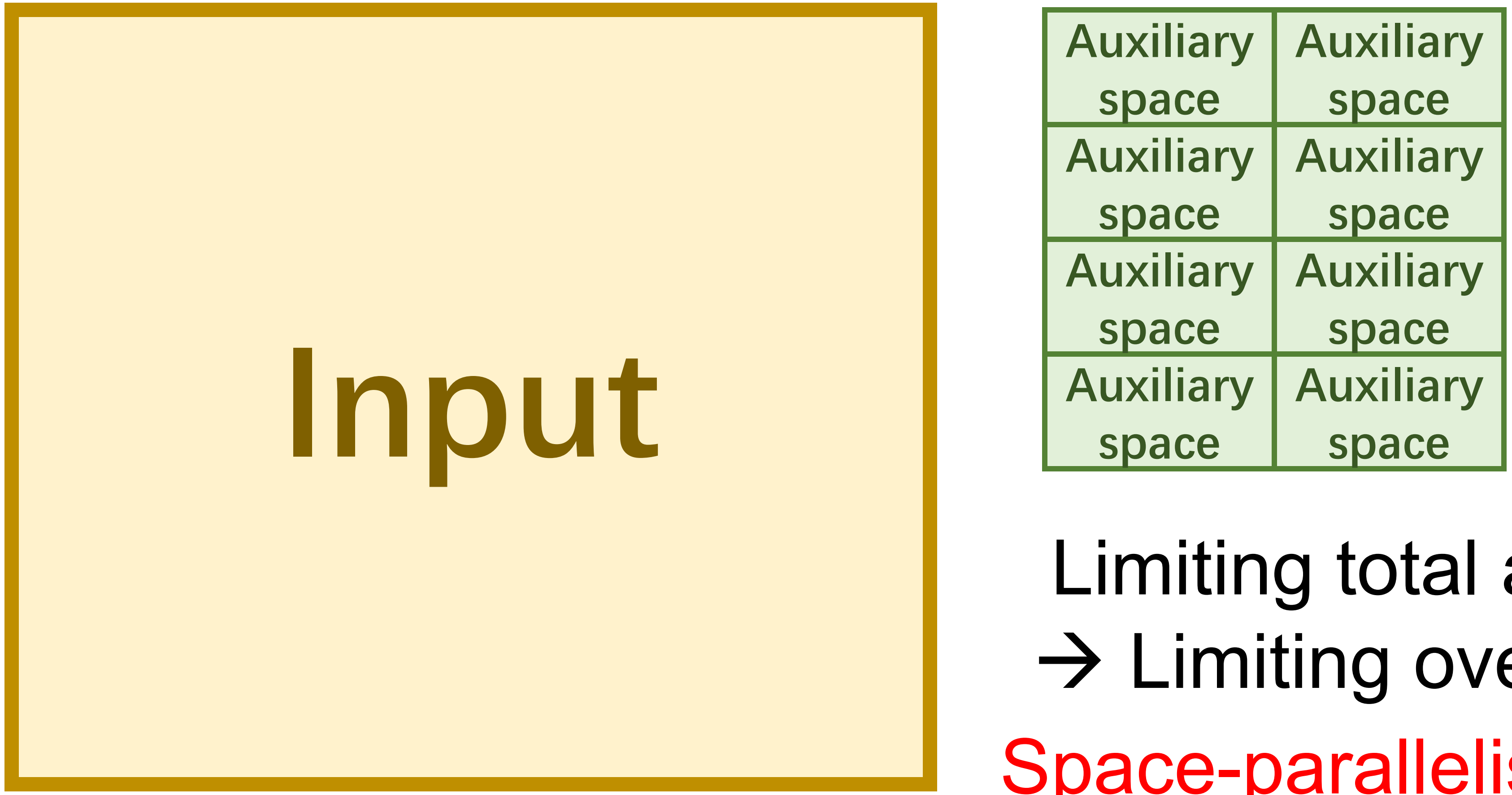
$$O(\mathrm{polylog}(n))$$

But it doesn’t quite work in the parallel setting…

Input

Auxiliary space	Auxiliary space	Auxiliary space	Auxiliary space
Auxiliary space	Auxiliary space	Auxiliary space	Auxiliary space
Auxiliary space	Auxiliary space	Auxiliary space	Auxiliary space
Auxiliary space	Auxiliary space	Auxiliary space	Auxiliary space
Auxiliary space	Auxiliary space	Auxiliary space	Auxiliary space
Auxiliary space	Auxiliary space	Auxiliary space	Auxiliary space
Auxiliary space	Auxiliary space	Auxiliary space	Auxiliary space
Auxiliary space	Auxiliary space	Auxiliary space	Auxiliary space

But it doesn't quite work in the parallel setting…



Limiting total auxiliary space
→ Limiting overall parallelism

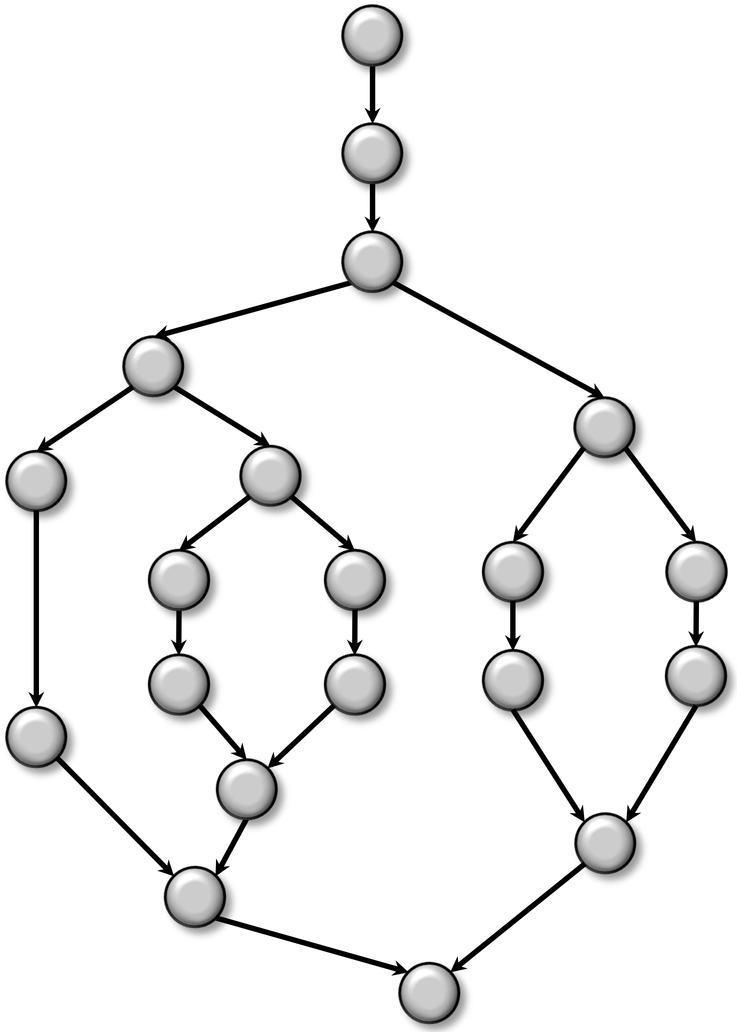
Space-parallelism tradeoff in the in-place PRAM model [Langston93]

Can we achieve both?

- **Can we get high parallelism?**
  - Low span
- **Can we achieve small auxiliary space?**
  - Each processor should use a small auxiliary space, similar to the sequential setting (e.g., $O(\log n)$ words)
- **Can we have clean computational models that capture both needs, but are still simple to use?**
  - Need to decouple the analysis of auxiliary space and the analysis of span

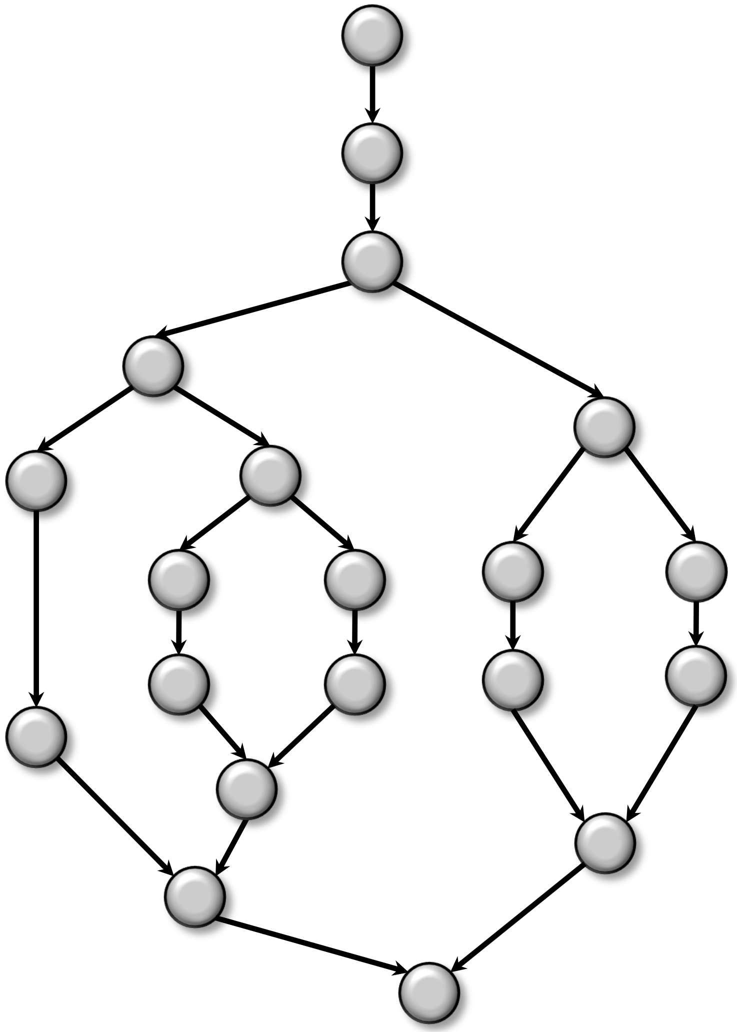
The binary fork-join (work-span) model

- An algorithm is measured by work (number of operations) and span (length of longest sequential dependence)
- A `fork` instruction creates two subtasks that can be run in parallel
- After they finish, they `join` and continue



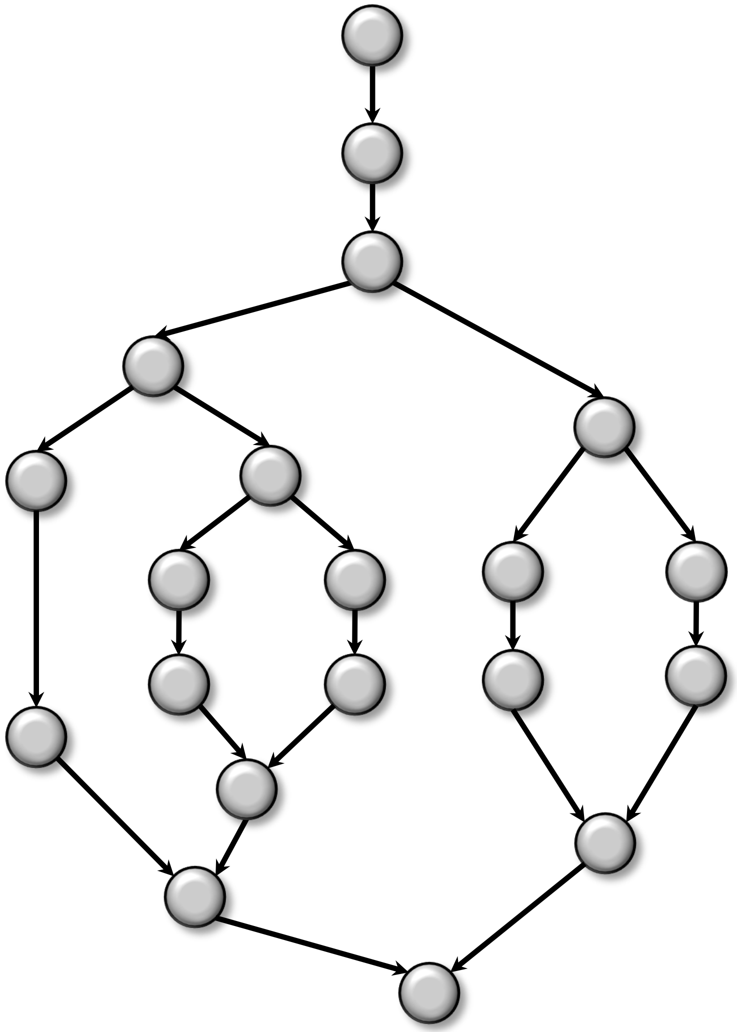
The binary fork-join (work-span) model

- **Benefits of this model:**
  - High-level, and algorithm designers need not to deal with system-level details such as load-balancing, task scheduling, synchronization, which are error-prone and can significantly complicate an algorithm
  - Algorithm design and analysis are independent of P (#processors)
- **Can we design parallel in-place (PIP) algorithm also based on this model?**



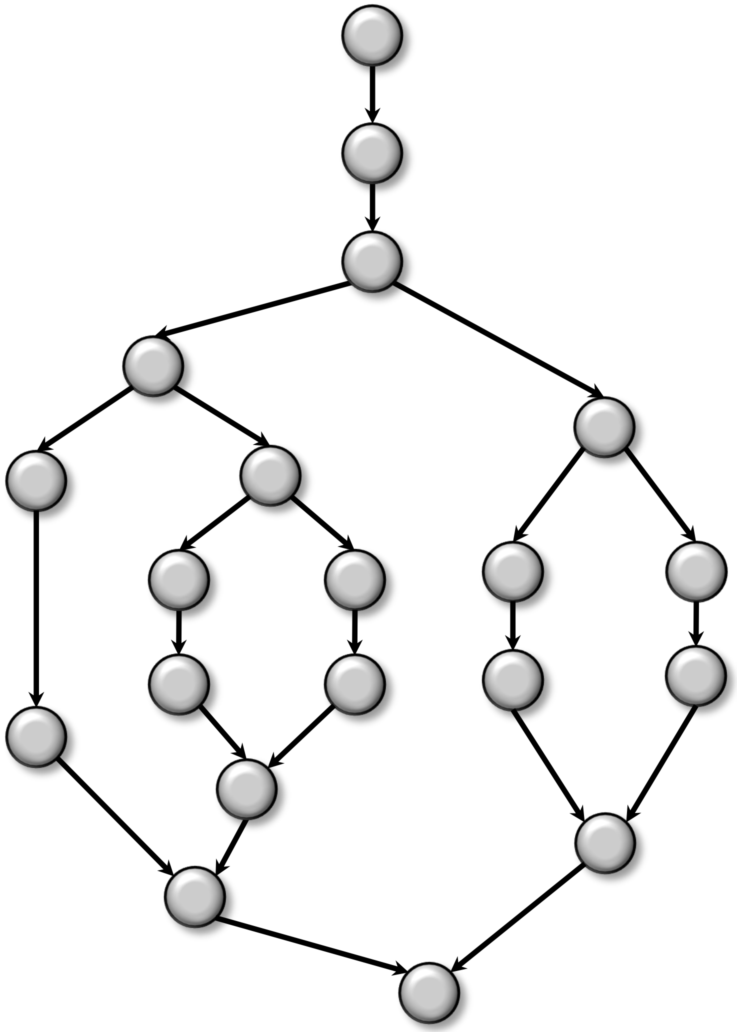
New models in this paper

- **Strong PIP model**
  - Achieve small (polylogarithmic) span and auxiliary space simultaneously
- **Relaxed PIP model**
  - Achieve sub-linear span and auxiliary space simultaneously
- **Our models decouple the analysis between span and auxiliary space**
  - Low span is useful in practice, not just for high parallelism, but also for reducing cache misses and global synchronization



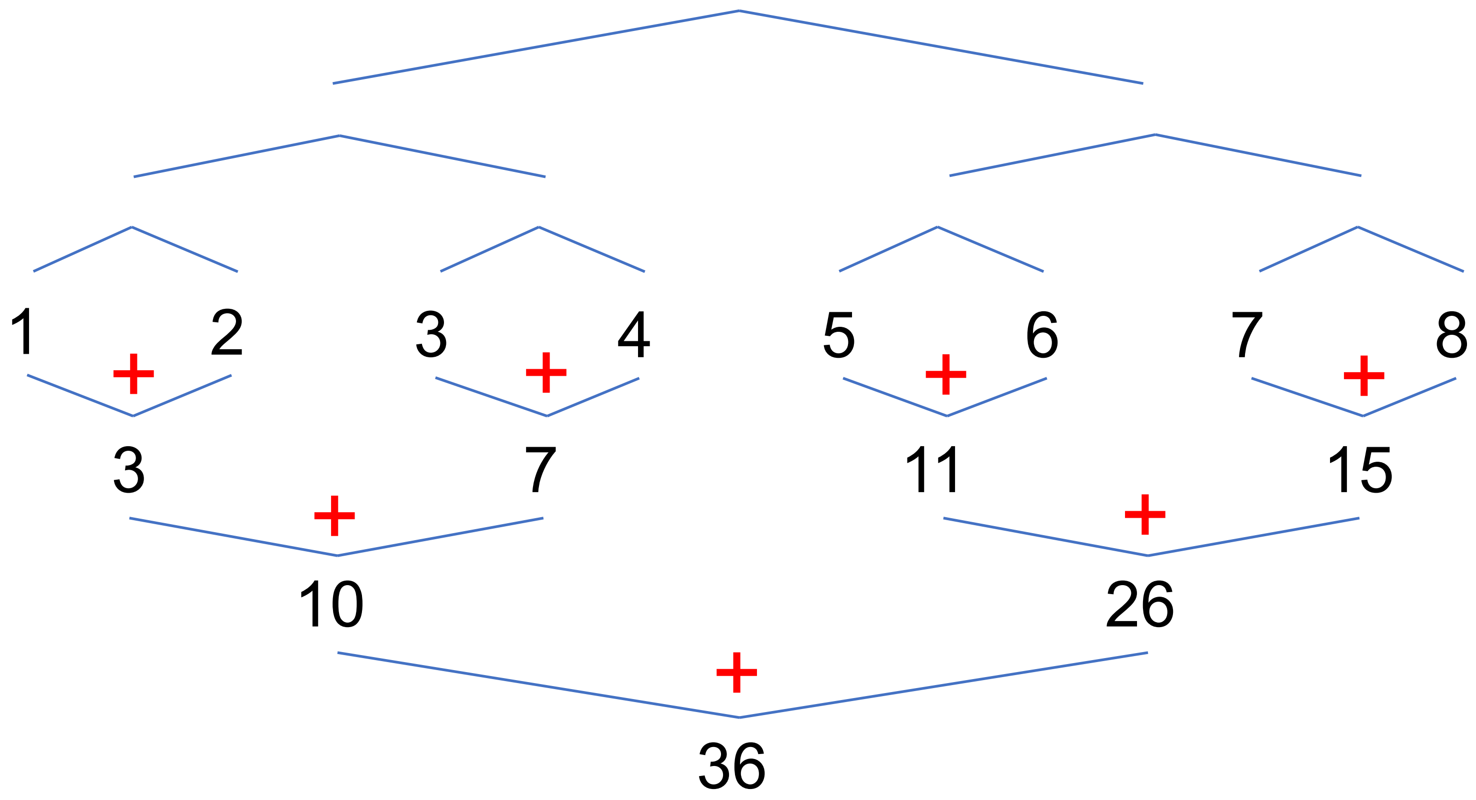
The strong PIP model

- **We assume:**
  - The sequential execution uses $O(\log n)$-word auxiliary space in a stack-allocated fashion for an input size of n
- **Stack-allocated fashion: when we allocate memory after a fork (or function call) it must be reclaimed before the associated join (or function return)**
- **A strong PIP algorithm uses $O(P \log n)$ total auxiliary space on P processors using a randomized work-stealing scheduler (e.g., Cilk)**
  - **The "busy-leaves" property [BL99]**



An strong PIP algorithm example

```
reduce(A, n) {
    if (n == 1) return A[0];
    In parallel:
        L = reduce(A, n/2);
        R = reduce(A+n/2, n-n/2);
    return L+R;
}
```



Work: $O(n)$

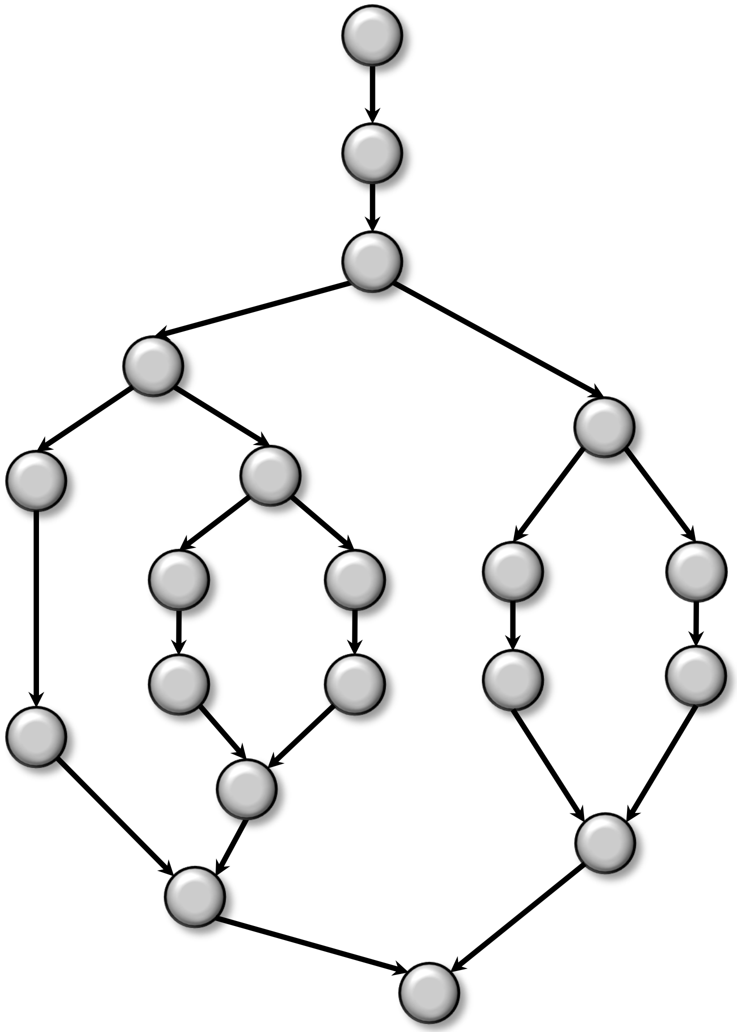
Span: $O(\log n)$

Sequential auxiliary space: $O(\log n)$

Total on P processors: $O(P \log n)$

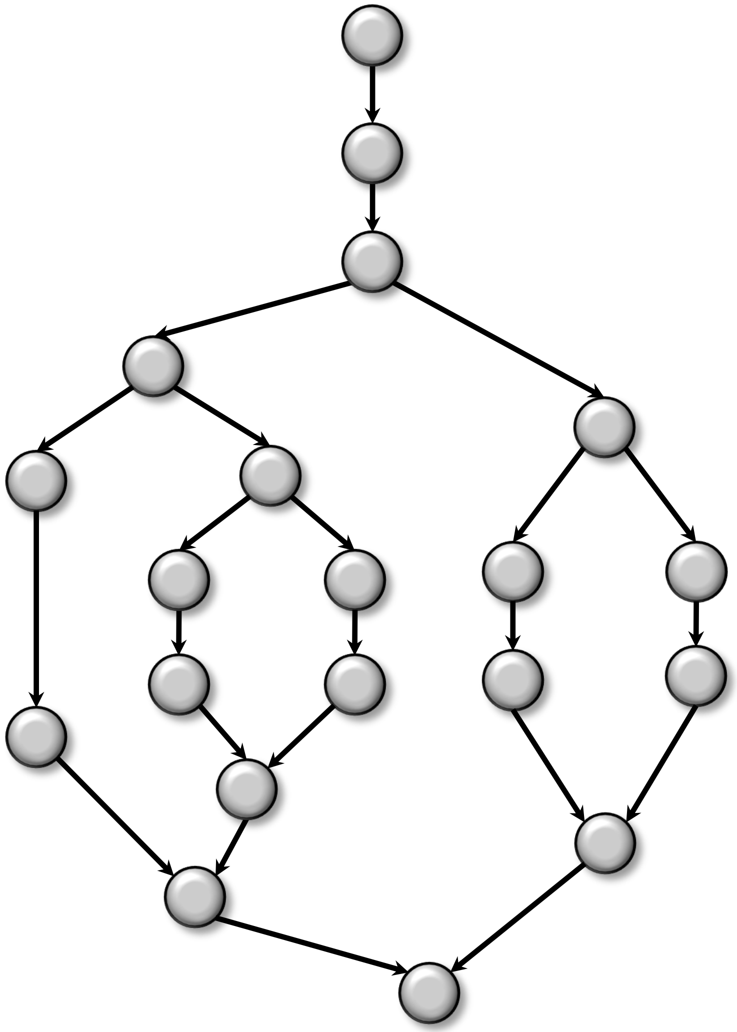
The strong PIP model

- **We assume:**
  - The sequential execution uses $O(\log n)$-word auxiliary space in a stack-allocated fashion for an input size of n
- **The strong PIP model is very restrictive**
  - Does not allow for heap space
  - We do not have many work-efficient PIP algorithms in this model



The relaxed PIP model

- **We assume:**
  - The sequential execution uses $O(\log n)$-word auxiliary space in a stack-allocated fashion, and $O(n^{\epsilon})$ shared (heap-allocated) auxiliary space $(0 < \epsilon < 1)$
- **Allows us to design many more work-efficient PIP algorithms**



Outline of this talk

1 Models for parallel in-place (PIP) algorithms

2 New PIP algorithms and a general approach

PIP algorithms

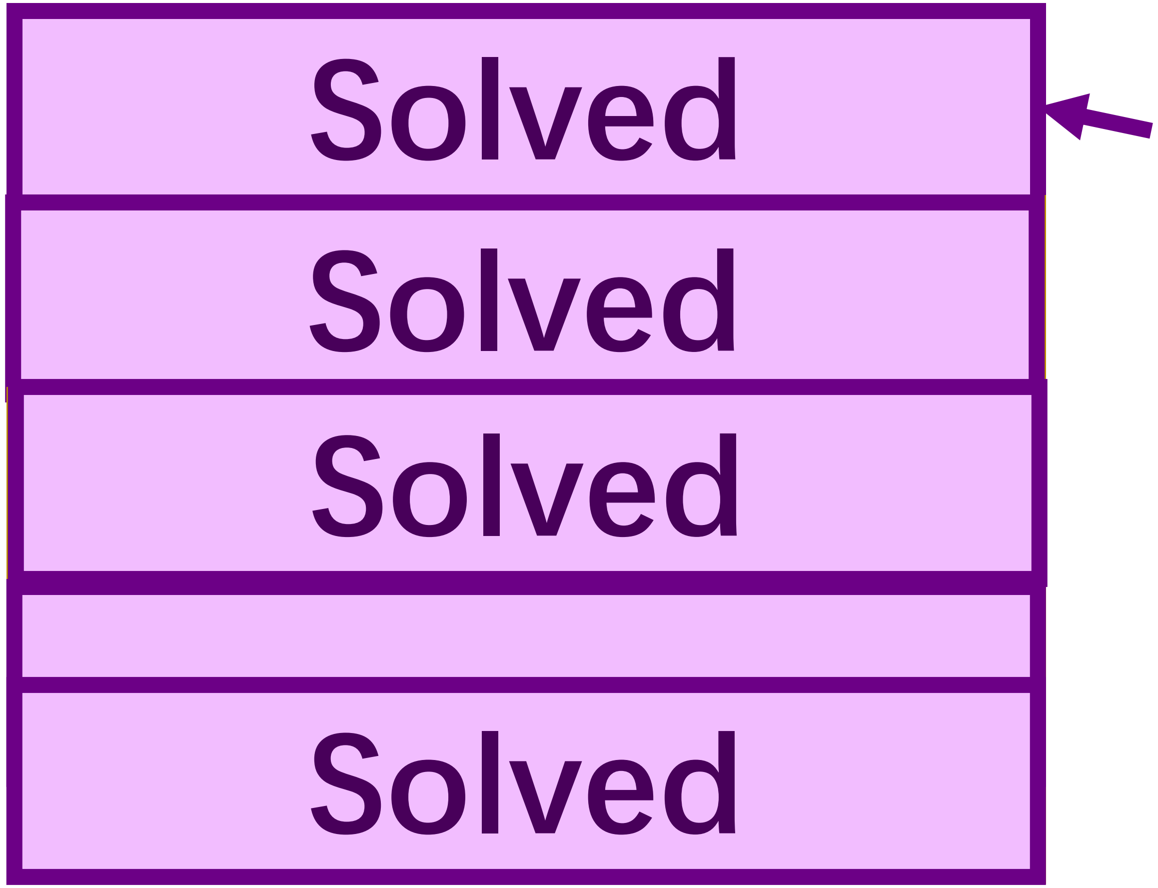
Model	Problems	Work-efficient	
Strong PIP Model	Permuting tree layout	✓	[6]
	Reduce, rotating	✓	
	Scan (prefix sum)	✓	*
	Filter, partition, quicksort		
	Merging, mergesort		
	Set operations	✓	[11]
Relaxed PIP Model	Random permutation	✓	*
	List and tree contraction	✓	*
	Merging, mergesort	✓	*
	Filter, partition, quicksort	✓	
	(Bi)Connectivity		*
	Minimum spanning forest		*

- **[6]: Berney et al., IPDPS 2019**
- **[11]: Blelloch, Ferizovic, Sun, SPAA 2016**

***: main contribution**

General approach for relaxed PIP algorithms

- **Decomposable Property**



Auxiliary space used is bounded by auxiliary space for sub-problem

Provide a tradeoff between **space** and **parallelism**

Relaxed PIP algorithm design using the Decomposable Property

- Suppose that there is an algorithm satisfying the decomposable property with work $W(n) = O\big(n \operatorname{polylog}(n)\big)$ and $O\big(\operatorname{polylog}(n)\big)$ span. Then, there is a relaxed PIP algorithm for the same problem with $W(n)$ work, $O\big(n^{\epsilon}\operatorname{polylog}(n)\big)$ span, and $O(n^{1-\epsilon})$ auxiliary space for some $0 < \epsilon < 1$.

Random Permutation as an example

KNUTHSHUFFLE(A, H)
  for $i \leftarrow n$ to 1 **do**
    swap($A[i]$, $A[H[i]]$)

$H[i]$ is randomly drawn between 1 and i

Iterate	1	2	3	4	5	6	7	8
H =	1	1	2	4	2	3	4	2
A =	a	b	c	d	e	f	g	h

This serial algorithm is in-place

This algorithm can be parallelized [SGB+15], with $O(n)$ work and $O(\log n)$ span w.h.p. [BFGS20]

However, the amount of auxiliary space is O(n), for data structures to resolve conflicts

Decomposable property

```
KnuthShuffle(A, H)
  for i ← n to 1 do
    swap(A[i], A[H[i]])
```

Iterate	1	2	3	4	5	6	7	8
H =	1	1	2	4	2	3	4	2
A =	a	b	c	d	e	f	g	h

Decomposable property

$\textsc{KnuthShuffle}(A, H)$
for $i \leftarrow n$ to 1 **do**
  swap$(A[i], A[H[i]])$

Iterate	1	2	3	4	5	6	7	8
H =	1	1	2	4	2	3	4	2
A =	a	b	c	d	e	f	g	h

Work on the second half first

Decomposable property

$\textsc{KnuthShuffle}(A, H)$
for $i \leftarrow n$ to 1 **do**
swap$(A[i], A[H[i]])$

Iterate	1	2	3	4	5	6	7	8
$H =$	1	1	2	4	2	3	4	2
$A =$	a	e	f	g	h	c	d	b

Work on the second half first

Decomposable property

$\textsc{KnuthShuffle}(A, H)$
for $i \leftarrow n$ to 1 **do**
swap($A[i], A[H[i]]$)

Iterate	1	2	3	4	5	6	7	8
$H =$	1	1	2	4	2	3	4	2
$A =$	a	e	f	g	h	c	d	b

Then work on the first half

Decomposable property

```
KNUTHSHUFFLE(A, H)
  for i ← n to 1 do
    swap(A[i], A[H[i]])
```

Iterate	1	2	3	4	5	6	7	8
H =	1	1	2	4	2	3	4	2
A =	a	b	c	d	e	f	g	h

Decomposable property

```
KnuthShuffle(A, H)
  for i ← n to 1 do
    swap(A[i], A[H[i]])
```

Iterate	1	2	3	4	5	6	7	8
H =	1	1	2	4	2	3	4	2
A =	a	b	c	d	e	f	g	h

Work on k elements per batch, for a total of n/k rounds

Only needs O(k) auxiliary space for resolving conflicts per round

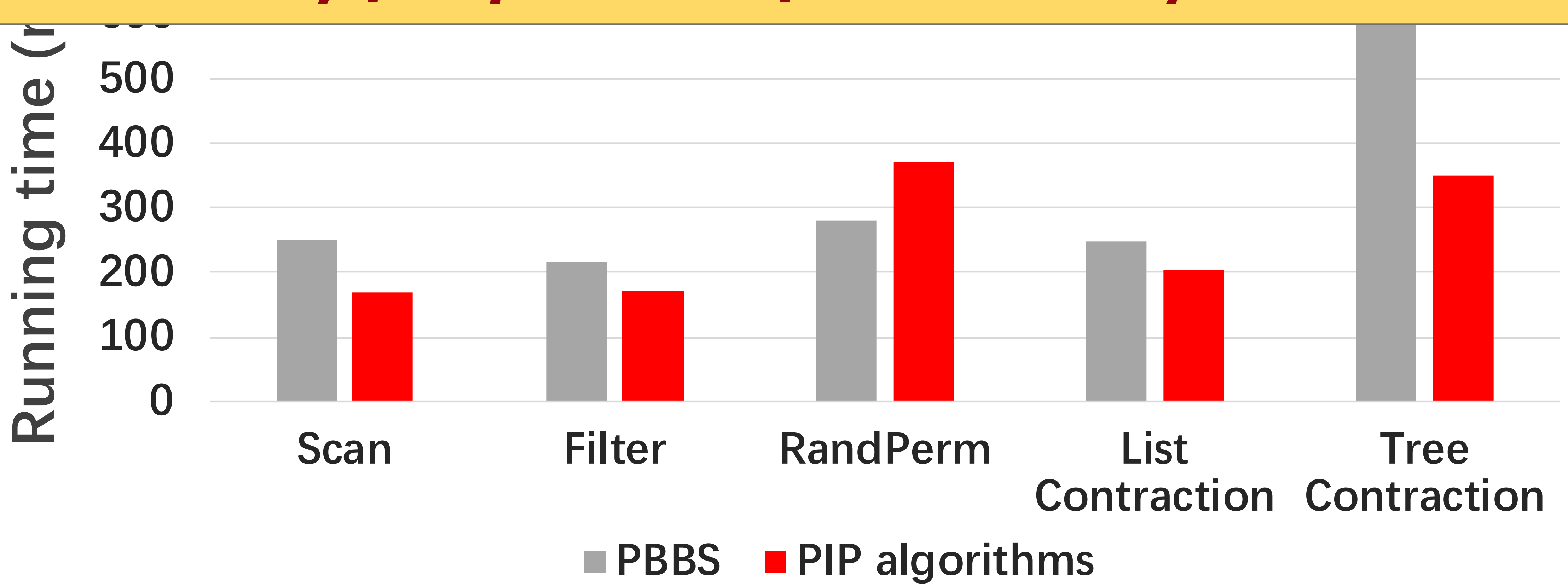
This gives an O(n) work relaxed PIP algorithm for random permutation, with sublinear span and space

Experiment setup

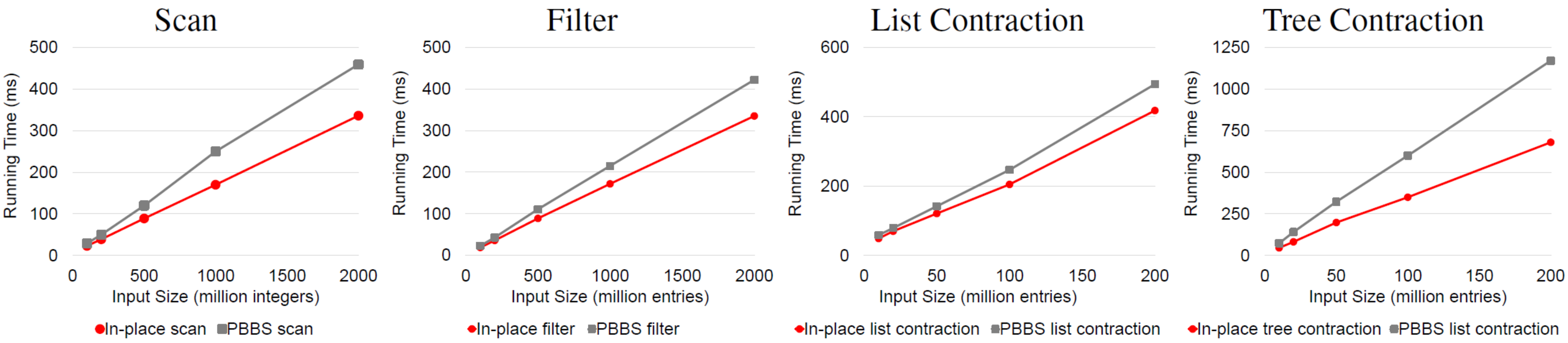
- 72-core Dell PowerEdge R930 (with two-way hyper-threading) and 1TB of main memory
- Implemented using Cilk Plus
- Comparing to Problem Based Benchmark Suite (PBBS), containing state-of-the-art multicore implementations

Overall running time

Our PIP algorithms are competitive with or faster than the best non-in-place versions, mainly due to a smaller memory footprint and fewer memory accesses.



Varying input sizes and thread counts



Our PIP algorithms have good scalability with respect to input size and thread counts, similar to the best non-in-place parallel algorithms.

Space Usage

- The PBBS algorithms are not in-place, and require auxiliary space linear in the input size
- Memory overhead of our PIP algorithms:

Problem	Input size (MB)	Memory usage (MB)	Over-head
Scan	7629.4	7636.2	<0.1%
Filter	7629.4	7636.9	<0.1%
Random permutation	762.9	791.2	3.7%
List contraction	762.9	773.5	1.4%
Tree contraction	1144.4	1154.9	0.9%

Summary

1 Models for parallel in-place (PIP) algorithms

Strong and relaxed PIP models, based on the binary fork-join model

Decouples the analysis between parallelism and auxiliary space, and leads to practical algorithms

2 New PIP algorithms and a general approach

Decomposable property: convert a non-PIP algorithm to relaxed PIP

New PIP algorithms for scan, filter, sort, merge, random permutation, list and tree contraction, (bi)connectivity, minimum spanning forest

Competitive with or faster than state-of-the-art in practice